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Choice of Boundary Conditions for Studying the Behavior of the Swarm of Spherical Particles Traveling through a Non-Newtonian Liquid

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Abstract—Possible boundary conditions for the cell model used in describing the behavior of a traveling swarm of spherical particles are considered. Gas–liquid systems are used to illustrate the effect of considered boundary conditions (Happel, Kuwabara, Slobodov–Chepura) on the calculated values of the components of the velocity field and the velocity of the swarm of bubbles traveling through a non-Newtonian disperse liquid medium.

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In calculating hydromechanical separation operations, such as flotation, sedimentation, and the like, one of the most important parameters is the velocity of the entire swarm of spherical dispersed particles, which can much differ from the velocities of single particles. For definiteness, from here on the spherical particles will be understood as the bubbles surrounded by a liquid and forming a gas–liquid system.

In addition to their size, the bubbles found in a gas–liquid disperse medium are characterized by a certain volume fraction, called the void fraction (gas holdup) φ. As it is noted in the literature [1, 2], when the concentration of dispersed particles is higher than 2 to 5 vol %, the disperse medium is characterized by a hindered flow, in which the separate particles (that is, the bubbles found in the gas–liquid system under consideration) begin acting on each other because of the interaction between the surrounding layers of the viscous liquid.

The cell model shown in Fig. 1 is widely used for the mathematical simulation of the regularities in the motion of a swarm of bubbles through a liquid [1, 3, 4]. According to this model, the bubbles are uniformly distributed throughout the liquid phase and every bubble is enclosed in the spherical cell formed by the disperse medium. The radius R of the liquid shell around the bubble is determined by the equality between the void fractions φ of the gaseous phase inside the cell and in the bulk gas–liquid medium. Using this equality, one can estimate the value of the outer cell radius:

$$R = R_b\varphi^{-1/3}. \quad (1)$$

This model adequately describes the limit transitions: $R \longrightarrow \infty$ as $\varphi \longrightarrow 0$ and $R \longrightarrow R_b$ as $\varphi \longrightarrow 1$.

The cell model can be used to reduce the solution of the boundary-value problem for the flow around a system of particles to the problem for a single particle (bubble). The difference consists in their boundary conditions. For a single particle, the outer boundary at which there is no change in the velocity field of the flow around the particle is found at infinity whereas the outer boundary for each particle in the system of particles is the shell of a cell with radius R, where the radius is determined by Eq. (1). The first of two necessary boundary conditions is the impermeability of the cell;

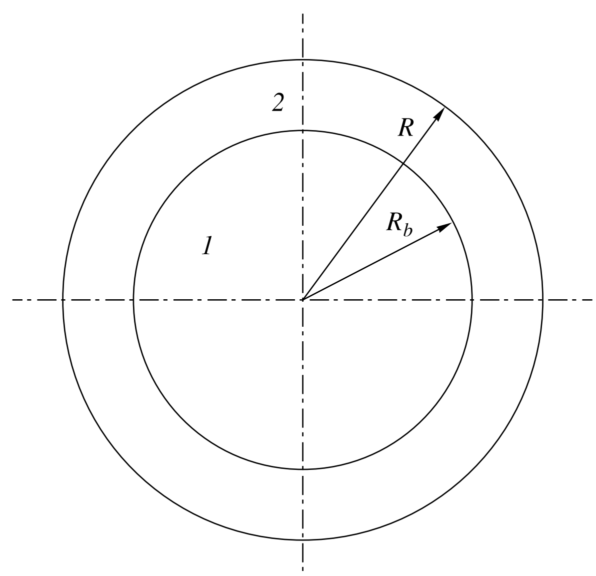

Fig. 1. Cell model: *1*, bubble; *2*, liquid shell.

that is, the normal component of the liquid velocity at the cell boundary vanishes to zero:

$$W_{r|r=R} = 0. \quad (2)$$

The above condition states that in the spherical system of coordinates ρ, ϑ, β attached to the center of the traveling bubble, the radial velocity of the liquid in the cell is equal to zero. At the same time, if we consider the physical picture of the motion of the bubble and its surrounding liquid relative to the external flow, it will be obvious that the continuous medium flows around this bubble and the cell rigidly attached to it as if they were one whole. In this case, the radial velocity component W_r at the cell boundary for the top pole, $\theta = 0$, is equal to the rise velocity of the bubble for the flow conditions under consideration or, which is equivalent in this case, the cell velocity $W = W_{r|r=R,\theta=0}$. The above was used in the study [5] to define the velocity W for a swarm of bubbles. The condition that the amount of liquid in the cell surrounding the bubble should be unchanged (cell impermeability) corresponds to the equation in which the flux of the curl of the radial velocity through the outer surface is equal to zero:

$$\iint_S W_r dS_{|r=R} = 0. \quad (3)$$

The cell impermeability condition formulated by the above expression is more accurate and physically correct.

There are various approaches to formulating the second boundary condition, each of which is based on certain assumptions (physical models) regarding the tangential velocity component W_θ and its derivatives at the outer boundary of the cell.

This condition was firstly introduced by Cunningham [6] in the form

$$W_{\theta|r=R\cdot} = 0. \quad (4)$$

The above condition combined with (2) postulates the absolute rigidity of the cell boundary. Actually, however, the outer cell boundary found in the bulk liquid should be a rather hypothetical surface and this condition is hardly correct from the hydrodynamic point of view.

A "less rigid" form of condition (4) at the outer cell boundary was proposed by Happel [7]:

$$\tau_{r\theta|r=R} = 0, \quad (5)$$

which can be equivalently rewritten in terms of velocity components as

$$\frac{1}{r}\frac{\partial W_r}{\partial\theta} + \frac{\partial W_\theta}{\partial r} - \frac{W_\theta}{r}_{|r=R} = 0. \quad (5a)$$

The above condition implies the existence of a free surface at the outer cell boundary or, in other words, the slip of the liquid relative to this boundary, which can be quite acceptable from the physical point of view.

A different approach, which was developed by Kuwabara [8], is based on the assumption that the curl of the velocity at the outer cell boundary is equal to zero:

$$rot\vec{W}_{|r=R} = 0, \quad (6)$$

or, in terms of velocity components,

$$-\frac{1}{r}\frac{\partial W_r}{\partial\theta} + \frac{\partial W_\theta}{\partial r} + \frac{W_\theta}{r}_{|r=R} = 0. \quad (6a)$$

The above condition is physically equivalent to the absence of closed circular flows (flow circulation) at the outer cell surface.

The Kuwabara condition was modified by Slobodov and Chepura [9], where the derivative of the curl of the velocity at the outer cell boundary rather than the curl itself was assumed to be equal to zero:

$$\frac{\partial}{\partial r}(rot\vec{W})_{|r=R} = 0, \quad (7)$$

or, in terms of velocity components,

$$\frac{1}{r^2}\frac{\partial W_r}{\partial\theta} - \frac{1}{r}\frac{\partial^2 W_r}{\partial r\partial\theta} + \frac{\partial^2 W_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial W_\theta}{\partial r} - \frac{W_\theta}{r^2}_{|r=R} = 0. \quad (7a)$$

Consequently, the above study assumes that the value of the curl of the velocity at the outer cell boundary should be constant, which includes a particular case in which its value is equal to zero. In this particular case, the modified condition becomes identical to the Kuwabara condition. The study [9] reports the results of calculating the velocity of the assemblage of solid particles traveling through a continuous medium using the proposed boundary condition and those proposed by Cunningham and Happel.

A comparison between the calculated results and experimental data showed that boundary condition (7) gives the best agreement. Unfortunately, the cited study does not report the comparison of the experimental data with the results calculated using the Kuwabara boundary condition. It could be attributed to the fact that the latter results were close to those calculated with condition (7) proposed by the authors of this study.

Analyzing the boundary conditions used in the cell model, it should be noted that although the formulas for the Happel and Kuwabara conditions are mathematically close to each other: the only difference is the signs before two terms [see formulas (5a) and (6a)], their hydrodynamic interpretations are much different. Obviously, each of them, as well as Slobodov–Chepura condition (7a), is physically meaningful and justified and can be used for the theoretical calculations of the velocity of a swarm of dispersed particles in a continuous medium.

Consider the application of the above boundary conditions to solving the problem of determining the velocity field and pressure in the flow of a non-Newtonian liquid around a swarm of bubbles. To describe the rheologic properties of this liquid, we will use the two-parameter power-law model, in which its effective viscosity μ_{ef} is given by the equation [1, 3]

$$\mu_{ef} = KE^{n-1}. \tag{8}$$

Based on the Levich approach, which was used in solving a similar problem for the flow of a Newtonian liquid around a single sphere (drop, bubble), the functions for the velocity components will be written as te following power series [10]:

$$\begin{aligned} W_r &= \left(\frac{b_1}{r^3} + \frac{b_2}{r} + b_3 + a_1 r^2\right)\cos\theta, \\ W_\theta &= \left(\frac{b_1}{2r^3} - \frac{b_2}{2r} - b_3 - 2a_1 r^2\right)\sin\theta, \\ P_\ell - P_{0\ell} &= KE^{n-1}\left(\frac{b_2}{r^2} + 10a_1 r\right)\cos\theta. \end{aligned} \tag{9}$$

To solve the formulated problem, we should also write analogous expressions for the velocities and pressure inside the bubble, denoting the respective components as W'_r, W'_θ, and P_b. Because the values of the velocities and pressure at the bubble center ($r \longrightarrow 0$) are bounded, the terms involving the variable r in the denominator should be omitted. As a result, the expressions for the above velocities take the form

$$\begin{aligned} W'_r &= (b'_3 + a'_1 r^2)\cos\theta, \\ W'_\theta &= -(b'_3 + 2a'_1 r^2)\sin\theta, \\ P_b - P_{0b} &= 10a'_1 r\mu_b\cos\theta. \end{aligned} \tag{10}$$

The problem formulated above is reducible to the determination of the values of unknown coefficients a_i and b_i using specific interface conditions and the condition at the bubble surface. In view of the physical picture of the flow under consideration, the used system of spherical coordinates will be attached to the center of the *traveling* bubble and the external flow will move relative to the bubble in the opposite direction at velocity $-W$. In this case, W is the velocity of the bubble itself in the fixed coordinate system.

The boundary conditions (that is, the values of velocity field components and pressure) will be formulated for the bubble surface, which is the interface, $r = R_b$, and for the outer cell boundary, $r = R$. At the interface, the radial velocity components of the non-Newtonian liquid flowing around the bubble, W_r, and the internal flow circulating in it, $-W'_r$, vanish to zero. At the same time, the values of tangential velocities W_θ and W'_θ and components of the tensor of viscous stresses, normal (τ_{rr}, τ'_{rr}) and shear $(\tau_{r\theta}, \tau'_{r\theta})$, remain continuous. Written in terms of velocities in view of expression (8) for the effective viscosity of a non-Newtonian liquid, these conditions take the form

$$W_{r|r=R_b} = 0, \tag{11}$$

$$W'_{r|r=R_b} = 0, \tag{12}$$

$$W_{\theta|r=R_b} = W'_{\theta|r=R_b}, \tag{13}$$

$$\begin{aligned} &P_b - P_{\ell|r=R_b} \\ &= 2\mu_b\frac{\partial W'_r}{\partial r} - 2KE^{n-1}\frac{\partial W_r}{\partial r} + (\rho_\ell - \rho_b)gR_b\cos\theta_{|r=R_b}, \end{aligned} \tag{14}$$

$$\begin{aligned} &KE^{n-1}\left(\frac{1}{r}\frac{\partial W_r}{\partial\theta} + \frac{\partial W_\theta}{\partial r} - \frac{W_\theta}{r}\right)_{|r=R_b} \\ &= \mu_b\left(\frac{1}{r}\frac{\partial W'_r}{\partial\theta} + \frac{\partial W'_\theta}{\partial r} - \frac{W'_\theta}{r}\right)_{|r=R_b}. \end{aligned} \tag{15}$$

In the right-hand side of Eq. (14), the term $\Delta\rho g R_b\cos\theta$, which accounts for the increase of the external hydrostatic pressure over the height of the bubble [when $\theta = 0$ at the top pole, $\cos\theta = 1$ and the value of P_ℓ is lower than that for $\theta = \pi$ at the bottom pole of the sphere, where P_ℓ is higher], is added to the pressure of the external liquid. The expression for the E-tensor of strain rates of the non-Newtonian liquid at the bubble surface, $r = R_b$, can be written in terms of velocity components as [3]

$$\begin{aligned} E = \Bigg[&2\left(\frac{\partial W_r}{\partial r}\right)^2 + 2\left(\frac{1}{r}\frac{\partial W_\theta}{\partial\theta}\right)^2 + 2\left(\frac{W_\theta}{r}\cot\theta\right)^2 \\ &+ \left(\frac{\partial W_\theta}{\partial r} - \frac{W_\theta}{r} + \frac{1}{r}\frac{\partial W_r}{\partial\theta}\right)^2\Bigg]^{\frac{1}{2}}. \end{aligned} \tag{16}$$

In the above expression, it is taken into account that $W_\beta \equiv 0$ because we assume that the flow around the bubble is symmetric with respect to the circumferential coordinate and $W_r = 0$ according to boundary condition (11).

The boundary condition at the cell surface will be written using one of the above three conditions (Happel, Kuwabara, or Slobodov-Chepura). We will consider the flow around a swarm of bubble at low (Re ≤ 1) and moderate (100 ≤ Re ≤ 1000) Reynolds numbers. The first case (Re ≤ 1) will be characterized by two flows, Stokes and Hadamard–Rybczynski, which correspond to the retarded and free surface of the bubble in the flow.

Happel condition. For the *Stokes flow*, the expression for the strain rate is given by $E = \left[\frac{\Delta\rho g R_b \sin\theta(1-\varphi^{5/3})}{(3+2\varphi^{5/3})K}\right]^{\frac{1}{n}}$. From here on, $\Delta\rho = \rho_\ell - \rho_b$. In this case, the coefficients in the equations given by (9) are written as

$$b_1 = \frac{\Delta\rho g R_b^5 E^{1-n}}{3(3+2\varphi^{5/3})K}, \quad b_2 = -\frac{\Delta\rho g}{3K} R_b^3 E^{1-n},$$

$$b_3 = \frac{\Delta\rho g R_b^2 (2+3\varphi^{5/3}) E^{1-n}}{3(3+2\varphi^{5/3})K}, \quad a_1 = -\frac{\Delta\rho g \varphi^{5/3} E^{1-n}}{3(3+2\varphi^{5/3})K}.$$

Among the parameters involved in the expression for E, the angular coordinate θ is variable and the others are constant. This implies that the strain rate varies over the surface of the bubble in the flow. In the calculations below, we will use the value of E^{1-n} averaged over the bubble surface S:

$$\bar{E}_{St}^{1-n} = \left[\frac{\Delta\rho g R_b (1-\varphi^{5/3})}{(3+2\varphi^{5/3})K}\right]^{\frac{1-n}{n}} I_{St}, \tag{17}$$

where

$$I_{St} = \frac{1}{4\pi R_b^2} \iint_S \sin^{\frac{1-n}{n}}\theta dS = \frac{1}{2}\int_0^\pi \sin^{\frac{1}{n}}\theta d\theta. \tag{18}$$

In this case, the values of the sought coefficients will be given by the formulas

$$b_1 = \left[\frac{\Delta\rho g (1-\varphi^{5/3})^{1-n}}{3^n(3+2\varphi^{5/3})K}\right]^{\frac{1}{n}} I_{St} R_b^{\frac{4n+1}{n}},$$

$$b_2 = -\left[\frac{\Delta\rho g}{3^n K}\left(\frac{1-\varphi^{5/3}}{3+2\varphi^{5/3}}\right)^{1-n}\right]^{\frac{1}{n}} I_{St} R_b^{\frac{2n+1}{n}},$$

$$b_3 = (2+3\varphi^{5/3})\left[\frac{\Delta\rho g (1-\varphi^{5/3})^{1-n}}{3^n(3+2\varphi^{5/3})K}\right]^{\frac{1}{n}} I_{St} R_b^{\frac{n+1}{n}},$$

$$a_1 = -\varphi^{5/3}\left[\frac{\Delta\rho g (1-\varphi^{5/3})^{1-n}}{3^n(3+2\varphi^{5/3})K}\right]^{\frac{1}{n}} I_{St} R_b^{\frac{1-n}{n}}.$$

Substitution of the above values of coefficients into Eq. (9) gives the desired expressions for the components of the field velocity and pressure:

$$W_{rSt} = \left[\frac{\Delta\rho g R_b^{n+1} (1-\varphi^{5/3})^{1-n}}{3^n(3+2\varphi^{5/3})K}\right]^{\frac{1}{n}} I_{St}$$
$$\times\left[\frac{R_b^3}{r^3} - (3+2\varphi^{5/3})\frac{R_b}{r} + (2+3\varphi^{5/3}) - \varphi^{5/3}\frac{r^2}{R_b^2}\right]\cos\theta,$$

$$W_{\theta St} = \left[\frac{\Delta\rho g R_b^{n+1} (1-\varphi^{5/3})^{1-n}}{3^n(3+2\varphi^{5/3})K}\right]^{\frac{1}{n}} I_{St} \tag{19}$$
$$\times\left[\frac{R_b^3}{2r^3} + (3+2\varphi^{5/3})\frac{R_b}{2r} - (2+3\varphi^{5/3}) + 2\varphi^{5/3}\frac{r^2}{R_b^2}\right]\sin\theta,$$

$$P_{St} = -\frac{\Delta\rho g}{3}\left[\frac{R_b^3}{r^2} + \frac{10\varphi^{5/3}}{3+2\varphi^{5/3}} r\right]\cos\theta.$$

The velocity of the swarm of spheres moving through a non-Newtonian liquid W_{St} can be determined by the value of the radial velocity component W_{rSt} at the top pole of the cell ($\theta = 0$, $r = R$) [5]:

$$W_{St} = \left[\left(\frac{2^n \Delta\rho g R_b^{n+1}}{3^n K}\right)\frac{(1-\varphi^{5/3})^{1-n}}{3+2\varphi^{5/3}}\right]^{\frac{1}{n}} I_{St} \tag{20}$$
$$\times\left(1 - \frac{3}{2}\varphi^{1/3} + \frac{3}{2}\varphi^{5/3} - \varphi^2\right).$$

When $n = 1$ $K \equiv \mu_\ell$, $I_{St} = 1$, and formulas (19) are transformed to the expressions for the components of the velocity field and pressure for the Stokes flow of a Newtonian liquid around a drop or bubble [4]. In this case, formula (20) becomes identical to the expression for the velocity of a swarm of bubbles with retarded surfaces obtained in the same study using the Happel boundary condition.

For the *Hadamard-Rybczynski* flow, the values of E and coefficients a_1, b_1 to b_3 are given by the expressions

$$E = \left(\frac{\Delta\rho g R_b \cos\theta}{3^{1/2} K}\right)^{\frac{1}{n}}$$

and

$$\bar{E}_{A-R}^{1-n} = \left(\frac{\Delta\rho g R_b}{3^{1/2} K}\right)^{\frac{1-n}{n}} I_{A-R},$$

where

$$I_{A-R} = \frac{1}{4\pi R_b^2}\iint_S \cos^{\frac{1-n}{n}}\theta dS = \frac{1}{2}\int_0^{\pi}\cos^{\frac{1-n}{n}}\theta\sin\theta d\theta, \tag{21}$$

$$a_1 = b_1 = 0, \quad b_2 = -\left(\frac{\Delta\rho g}{3^{\frac{n+1}{2}}K}\right)^{\frac{1}{n}} I_{A-R} R_b^{\frac{2n+1}{n}},$$

$$b_3 = \left(\frac{\Delta\rho g}{3^{\frac{n+1}{2}}K}\right)^{\frac{1}{n}} I_{A-R} R_b^{\frac{n+1}{n}}.$$

The above expressions for the coefficients b_2 and b_3 and the parameter E do not involve the parameter φ, the void fraction in the gas–liquid system. Mathematically, this can be attributed to the fact that the coefficients a_1 and b_1, which involve the parameter φ, are equal to zero. Physically, when the surface of the sphere (bubble) is free and the shear stresses at the cell boundary are equal to zero, the flow of the liquid in the cell remains geometrically similar no matter what its size is. This implies that as the value of parameter φ varies (for definiteness, φ will decrease, which is equivalent to the increasing size of the cell), the distance between the adjacent streamlines of the liquid flowing around the bubble becomes larger, which scales up the flow pattern in the cell, retaining the same geometric configuration of the streamlines. As $\varphi \longrightarrow 0$, the limit transition to the flow around a single bubble takes place. At the same time, the value of the velocity of the swarm of bubbles for the Hadamard-Rybczynski flow W_{A-R}, which is likewise (as in Stokes flow) equal to the flow velocity relative to the top pole of the cell, will be given by the expression

$$W_{A-R} = \left(\frac{\Delta\rho g R_b^{n+1}}{3^{\frac{n+1}{2}}K}\right)^{\frac{1}{n}} I_{A-R}(1-\varphi^{1/3}), \tag{22}$$

which depends on the void fraction in the gas–liquid system.

The necessary condition for the existence of potential flow is the irrotational behavior of the liquid flow, including the flow at the cell boundary, which corresponds to the Kuwabara condition (see below). Consequently, the Happel condition cannot be used for this flow mode.

Kuwabara condition. For the *Stokes flow*, the tensor of strain rates and the coefficients in the expressions for the components of the velocity field and pressure for a non-Newtonian liquid are written as

$$E = \left[\frac{\Delta\rho g R_b \sin\theta(1-\varphi)}{3K}\right]^{\frac{1}{n}},$$

$$\bar{E}_{St}^{1-n} = \left[\frac{\Delta\rho g R_b(1-\varphi)}{3K}\right]^{\frac{1-n}{n}} I_{St},$$

$$b_1 = \left[\frac{\Delta\rho g(1-\varphi)^{1-n}}{3K}\right]^{\frac{1}{n}} I_{St}\left(\frac{1}{3}-\frac{2\varphi}{15}\right)R_b^{\frac{4n+1}{n}},$$

$$b_2 = -\left[\frac{\Delta\rho g(1-\varphi)^{1-n}}{3K}\right]^{\frac{1}{n}} I_{St} R_b^{\frac{2n+1}{n}},$$

$$b_3 = \left[\frac{\Delta\rho g(1-\varphi)^{1-n}}{3K}\right]^{\frac{1}{n}} I_{St}\left(\frac{2}{3}+\frac{\varphi}{3}\right)R_b^{\frac{n+1}{n}},$$

$$a_1 = -\frac{\varphi}{5}\left[\frac{\Delta\rho g(1-\varphi)^{1-n}}{3K}\right]^{\frac{1}{n}} I_{St} R_b^{\frac{1-n}{n}}.$$

The above expressions combined with Eqs. (9) can be used to derive the expressions for the components of the velocity field and pressure for the flow of a non-Newtonian liquid around an assemblage of spheres:

$$\begin{aligned} W_{rSt} &= \left[\frac{\Delta\rho g(1-\varphi)^{1-n}}{3K}\right]^{\frac{1}{n}} I_{St} \\ &\times\left[\left(\frac{1}{3}-\frac{2\varphi}{15}\right)\frac{R_b^3}{r^3}-\frac{R_b}{r}+\left(\frac{2}{3}+\frac{\varphi}{3}\right)-\frac{\varphi}{5}\frac{r^2}{R_b^2}\right]\cos\theta, \\ W_{\theta St} &= \left[\frac{\Delta\rho g(1-\varphi)^{1-n}}{3K}\right]^{\frac{1}{n}} I_{St} \\ &\times\left[\left(\frac{1}{6}-\frac{\varphi}{15}\right)\frac{R_b^3}{r^3}+\frac{R_b}{2r}-\left(\frac{2}{3}+\frac{\varphi}{3}\right)+\frac{2\varphi}{5}\frac{r^2}{R_b^2}\right]\sin\theta, \\ P_{St} &= -\frac{\Delta\rho g}{3}\left[\frac{R_b^3}{r^2}+2\varphi r\right]\cos\theta. \end{aligned} \tag{23}$$

When the Kuwabara condition is used, the velocity of an assemblage of spheres in a non-Newtonian liquid W_{St}, which is defined as before, is given by the expression

$$\begin{aligned} W_{St} &= \left[\frac{2^n\Delta\rho g R_b^{n+1}(1-\varphi)^{1-n}}{3^{n+1}K}\right]^{\frac{1}{n}} I_{St} \\ &\times\left(1-\frac{9}{5}\varphi^{1/3}+\varphi-\frac{\varphi^2}{5}\right). \end{aligned} \tag{24}$$

For the *Hadamard-Rybczynski* flow around an assemblage of spheres, the expressions for E and coefficients a_1, b_1 to b_3 can be written as

$$E = \left(\frac{\Delta\rho g R_b \cos\theta(1-\varphi)}{3^{1/2}K}\right)^{\frac{1}{n}},$$

$$\text{и } \bar{E}_{A-R}^{1-n} = \left(\frac{\Delta\rho g R_b(1-\varphi)}{3^{1/2}K}\right)^{\frac{1-n}{n}} I_{A-R},$$

$$b_1 = \frac{\varphi}{5}\left[\frac{\Delta\rho g(1-\varphi)^{1-n}}{3^{\frac{n+1}{2}}K}\right]^{\frac{1}{n}} I_{A-R} R_b^{\frac{4n+1}{n}},$$

$$b_2 = -\left(\frac{\Delta\rho g(1-\varphi)^{1-n}}{3^{\frac{n+1}{2}}K}\right)^{\frac{1}{n}} I_{A-R} R_b^{\frac{2n+1}{n}},$$

$$b_3 = \left(\frac{\Delta\rho g(1-\varphi)^{1-n}}{3^{\frac{n+1}{2}}K}\right)^{\frac{1}{n}} I_{A-R} R_b^{\frac{n+1}{n}},$$

$$a_1 = -\frac{\varphi}{5}\left[\frac{\Delta\rho g(1-\varphi)^{1-n}}{3^{\frac{n+1}{2}}K}\right]^{\frac{1}{n}} I_{A-R} R_b^{\frac{1-n}{n}}.$$

The components of the velocity field and the pressure for the Hadamard-Rybczynski flow of a non-Newtonian liquid are given by the equations

$$W_{rA-R} = \left[\frac{\Delta\rho g(1-\varphi)^{1-n}}{3^{\frac{n+1}{2}}K}\right]^{\frac{1}{n}} I_{A-R}$$

$$\times\left(\frac{\varphi}{5}\frac{R_b^3}{r^3} - \frac{R_b}{r} + 1 - \frac{\varphi}{5}\frac{r^2}{R_b^2}\right)\cos\theta,$$

$$W_{\theta A-R} = \left[\frac{\Delta\rho g(1-\varphi)^{1-n}}{3^{\frac{n+1}{2}}K}\right]^{\frac{1}{n}} I_{A-R} \quad (25)$$

$$\times\left(\frac{\varphi}{10}\frac{R_b^3}{r^3} + \frac{R_b}{2r} - 1 + \frac{2\varphi}{5}\frac{r^2}{R_b^2}\right)\sin\theta,$$

$$P_{St} = -\frac{\Delta\rho g}{3}\left(\frac{R_b^3}{r^2} + 2\varphi r\right)\cos\theta.$$

For the Hadamard-Rybczynski flow, the expression for the velocity of an assemblage of spheres W_{A-R} found using the Kuwabara boundary condition takes the form

$$W_{A-R} = \left(\frac{\Delta\rho g R_b^{n+1}}{3^{\frac{n+1}{2}}K}\right)^{\frac{1}{n}}(1-\varphi)^{\frac{1-n}{n}} I_{A-R} \quad (26)$$

$$\times\left(1 - \frac{6}{5}\varphi^{1/3} + \frac{\varphi^2}{5}\right).$$

For the *potential flow* around a swarm of bubbles, the liquid flow can be regarded as ideal. In this case, the dissipation of energy due to the viscous friction in the bulk continuous medium is ignored. In contrast to the Stokes and Hadamard-Rybczynski flows considered above, the surface of bubbles in the potential flow becomes "definite" and can therefore be regarded as free (moving). In this case, the effect of the viscous properties of the liquid is localized within the thin boundary layer near the bubble surface. This assumption makes it possible to introduce and use the model of ideal fluid, in which the flow of the fluid outside the boundary layer is irrotational, curl ($\mathrm{rot}_\beta \vec{W}$ I = 0 [10]. In this case, the components of the velocity field W_{rP} and $W_{\theta P}$ can be written using the corresponding derivatives of the stream function ψ that satisfies the Laplace equation $\Delta\psi = 0$. The latter written in terms of spherical coordinates is given by the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) = 0. \quad (27)$$

The general solution to the above equation is $\psi = \left(\frac{a}{r^2} + br\right)\cos\theta$, from which we can express the components of the velocity field W_{rP} and $W_{\theta P}$:

$$W_{rP} = \frac{\partial\psi}{\partial r} = \left(b - \frac{2a}{r^3}\right)\cos\theta,$$

$$W_{\theta P} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = -\left(b + \frac{a}{r^3}\right)\sin\theta.$$

The constant coefficients a and b, which are involved in the general solution of Eq. (27) and in the expressions for the components of the velocity field W_{rP} and $W_{\theta P}$, can be determined using the boundary conditions, which are the values of the radial velocity at the surface of the bubble and the outer surface of the cell enclosing the bubble:

$$W_{rP|r = R_b} = 0, \quad W_{rP|r = R, \theta = 0} = W_P,$$

where W_P is the velocity of the external potential flow of the liquid relative to the top pole of the cell. Substitution of the values of a and b obtained from the boundary conditions into the expressions for the velocity components W_{rp} and $W_{\theta P}$ gives the formulas

$$W_{rP} = \frac{W_P}{1-\varphi}\left(1 - \frac{R_b^3}{r^3}\right)\cos\theta,$$
$$W_{\theta P} = -\frac{W_P}{1-\varphi}\left(1 + \frac{R_b^3}{2r^3}\right)\sin\theta. \quad (28)$$

The distribution of the velocities near the surface of the sphere, $r \approx R_b$, where the effect of viscous forces should be taken into account, will be found by introducing a new variable $y = R_b - r$ and expanding expression (28) into a power series in y/R_b [10]:

$$W_{rP|r\approx R_b} = -3\frac{W_P}{(1-\varphi)}\frac{y}{R_b}\cos\theta,$$
$$W_{\theta P|r\approx R_b} = \frac{3}{2}\frac{W_P}{(1-\varphi)}\left(1 - \frac{y}{R_b}\right)\sin\theta. \quad (29)$$

The amount of the energy dissipated in the boundary layer, where the effect of viscous forces is localized, can be written as

$$-\frac{dU}{dt_{|r\approx R_b}} = KE^{n-1}\iint_S \frac{\partial \vec{W}^2_{|r\approx R_b}}{\partial \vec{n}}dS. \quad (30)$$

Here, $\frac{\partial \vec{W}^2_{|r\approx R_b}}{\partial \vec{n}} = \frac{\partial}{\partial r_{|r\approx R_b}}(W_{rP}^2 + W_{\theta P}^2)$ is the partial derivative of the velocity vector of the external flow near the surface of the sphere in the flow, $dS = R_b^2 \sin\theta d\theta d\beta$ is the elementary surface area of the sphere.

The value of E involved in Eq. (27) is determined by formula (16) using the components of the velocity field defined by formulas (28). Using these formulas, we obtain

$$-\frac{dU}{dt_{|r\approx R_b}} = 3^{\frac{3n+1}{2}}\pi K\left(\frac{W_P}{1-\varphi}\right)^{n+1} R_b^{2-n} I_P, \quad (31)$$

where

$$I_P = \int_0^\pi \cos^{n-1}\theta \sin^3\theta d\theta \quad (\text{for } n = 1\ I_P = 4/3), \quad (32)$$

In this case, the total dissipative force (drag) F_d will be given by the expression:

$$F_d = \frac{3^{\frac{3n+1}{2}}(n+1)}{2}\left(\frac{1}{1-\varphi}\right)^{n+1}\pi K W_P^n R_b^{2-n} I_P. \quad (33)$$

Writing the equation in which the drag is equal to the buoyancy (Archimedes) force, we can obtain the expression for the velocity of an assemblage of spheres relative to the potential flow of a non-Newtonian liquid:

$$W_P = \left[\frac{8\Delta\rho g(1-\varphi)^{n+2} R_b^{n+1}}{27^{\frac{n+1}{2}}(n+1)I_P K}\right]^{\frac{1}{n}}. \quad (34)$$

When $n = 1$ and $\varphi = 0$, expressions (23), (24) and (25), (26) are transformed to the respective formulas describing the Stokes and Hadamard-Rybczynski flows around a single particle [1, 11]. When $n = 1$, expressions (28) for the components of the velocity field of the potential flow of a non-Newtonian liquid around an assemblage of spheres become identical to those reported for a Newtonian liquid [5]. In that paper, however, the amount of dissipated energy was determined without consideration of the velocity components near the sphere surface, $r \approx R_b$, where the effect of viscous forces manifests itself, as it should be done according to Levich's approach used in that paper. In addition, rules [5] did not consider the decrease of the buoyancy force acting on the bubble by a factor of $(1 - \varphi)$ due to the presence of bubbles in the system. As a result, the formula for the velocity of the flow of a Newtonian liquid around a swarm of bubbles obtained in that paper differs from formula (34) derived in the present paper and is thought to be inaccurate.

Slobodov–Chepura condition. For the *Stokes flow* around a swarm of bubbles, the expressions for E and coefficients a_1, b_1 to b_3 take the form

$$E = \left[\frac{\Delta\rho g R_b \sin\theta(1+2\varphi)}{3K}\right]^{\frac{1}{n}},$$

and

$$\bar{E}_{St}^{\frac{1-n}{n}} = \left[\frac{\Delta\rho g R_b(1+2\varphi)}{3K}\right]^{\frac{1-n}{n}} I_{St},$$

$$b_1 = \left[\frac{\Delta\rho g(1+2\varphi)^{1-n}}{3K}\right]^{\frac{1}{n}}\left(\frac{1}{3} + \frac{4\varphi}{15}\right) I_{St} R_b^{\frac{4n+1}{n}},$$

$$b_2 = -\left[\frac{\Delta\rho g(1+2\varphi)^{1-n}}{3K}\right]^{\frac{1}{n}} I_{St} R_b^{\frac{2n+1}{n}},$$

$$b_3 = \left[\frac{\Delta\rho g(1+2\varphi)^{1-n}}{3K}\right]^{\frac{1}{n}} I_{St}\left(\frac{2}{3} - \frac{2\varphi}{3}\right) R_b^{\frac{n+1}{n}},$$

$$a_1 = -\frac{2\varphi}{5}\left[\frac{\Delta\rho g(1+2\varphi)^{1-n}}{3K}\right]^{\frac{1}{n}} I_{St} R_c^{\frac{1-n}{n}}.$$

The expressions for the components of the velocity field and pressure in the flow of a non-Newtonian liquid around an assemblage of spheres are written as

$$W_{rSt} = \left[\frac{\Delta\rho g(1+2\varphi)^{1-n}}{3K}\right]^{\frac{1}{n}} I_{St}$$
$$\times\left[\left(\frac{1}{3}+\frac{4\varphi}{15}\right)\frac{R_b^3}{r^3}-\frac{R_b}{r}+\frac{2}{3}(1-\varphi)+\frac{2\varphi}{5}\frac{r^2}{R_b^2}\right]\cos\theta,$$
$$W_{\theta St} = \left[\frac{\Delta\rho g(1+2\varphi)^{1-n}}{3K}\right]^{\frac{1}{n}} I_{St} \tag{35}$$
$$\times\left[\left(\frac{1}{6}+\frac{2\varphi}{15}\right)\frac{R_b^3}{r^3}+\frac{R_b}{2r}-\frac{2}{3}(1-\varphi)-\frac{4\varphi}{5}\frac{r^2}{R_b^2}\right]\sin\theta,$$
$$P_{St} = -\frac{\Delta\rho g}{3}\left(\frac{R_b^3}{r^2}-4\varphi r\right)\cos\theta.$$

When the Slobodov–Chepura condition is used, the velocity of the assemblage of spheres in the flow of a non-Newtonian liquid is given by the relation

$$W_{St} = \left[\frac{2^n\Delta\rho g R_b^{n+1}(1+2\varphi)^{1-n}}{3^{n+1}K}\right]^{\frac{1}{n}} I_{St} \tag{36}$$
$$\times\left(1-\frac{9}{10}\varphi^{1/3}-\frac{\varphi}{2}+\frac{2\varphi^2}{5}\right).$$

For the *Hadamard-Rybczynski* flow, the expressions for E and coefficients a_1, b_1 to b_3 take the form

$$E = \left(\frac{\Delta\rho g R_b\cos\theta(1+2\varphi)}{3^{1/2}K}\right)^{\frac{1}{n}},$$
$$\bar{E}_{A-R}^{1-n} = \left(\frac{\Delta\rho g R_b(1+2\varphi)}{3^{1/2}K}\right)^{\frac{1-n}{n}} I_{A-R},$$
$$b_1 = -\frac{2\varphi}{5}\left[\frac{\Delta\rho g(1+2\varphi)^{1-n}}{3^{\frac{n+1}{2}}K}\right]^{\frac{1}{n}} I_{A-R}R_b^{\frac{4n+1}{n}},$$
$$b_2 = -\left(\frac{\Delta\rho g(1+2\varphi)^{1-n}}{3^{\frac{n+1}{2}}K}\right)^{\frac{1}{n}} I_{A-R}R_b^{\frac{2n+1}{n}},$$
$$b_3 = \left(\frac{\Delta\rho g(1+2\varphi)^{1-n}}{3^{\frac{n+1}{2}}K}\right)^{\frac{1}{n}} I_{A-R}R_b^{\frac{n+1}{n}},$$
$$a_1 = \frac{2\varphi}{5}\left[\frac{\Delta\rho g(1+2\varphi)^{1-n}}{3^{\frac{n+1}{2}}K}\right]^{\frac{1}{n}} I_{A-R}R_b^{\frac{1-n}{n}}.$$

For the non-Newtonian flow considered, the components of the velocity field and pressure are given by the equations

$$W_{rA-R} = \left[\frac{\Delta\rho g(1+2\varphi)^{1-n}}{3^{\frac{n+1}{2}}K}\right]^{\frac{1}{n}} I_{A-R}$$
$$\times\left(-\frac{2\varphi}{5}\frac{R_b^3}{r^3}-\frac{R_b}{r}+1+\frac{2\varphi}{5}\frac{r^2}{R_b^2}\right)\cos\theta,$$
$$W_{\theta A-R} = \left[\frac{\Delta\rho g(1+2\varphi)^{1-n}}{3^{\frac{n+1}{2}}K}\right]^{\frac{1}{n}} I_{A-R} \tag{37}$$
$$\times\left(-\frac{\varphi}{5}\frac{R_b^3}{r^3}+\frac{R_b}{2r}-1-\frac{4\varphi}{5}\frac{r^2}{R_b^2}\right)\sin\theta,$$
$$P_{St} = -\frac{\Delta\rho g}{3}\left(\frac{R_b^3}{r^2}-4\varphi r\right)\cos\theta.$$

The the velocity of the assemblage of spheres W_{A-R} found using the Slobodov–Chepura boundary condition takes the form

$$W_{A-R} = \left(\frac{\Delta\rho g R_b^{n+1}}{3^{\frac{n+1}{2}}K}\right)^{\frac{1}{n}}(1+2\varphi)^{\frac{1-n}{n}} I_{A-R} \tag{38}$$
$$\times\left(1-\frac{3}{5}\varphi^{1/3}-\frac{2\varphi^2}{5}\right).$$

Integrals (18), (21), (32), which depend on the flow behavior index n, were evaluated numerically (Fig. 2). Figures 3 to 5 illustrate the calculated results for the flow velocity of a non-Newtonian liquid ($K = 0.01$ Pa c^n, where n is varied from 0.6 to 1.2) relative to a swarm of bubbles at different values of void fraction φ in the gas–liquid system for the Stokes, Hadamard-Rybczynski, and potential flows. The dotted lines in the graphs indicate the portions of the curves where the sizes of bubbles are beyond the limits of the corresponding flow modes.

It can be seen from the graphs that the void fraction in the gas–liquid system strongly affects the relative velocity of the swarm of bubbles. For the Stokes and Hadamard-Rybczynski flows, the value of this velocity

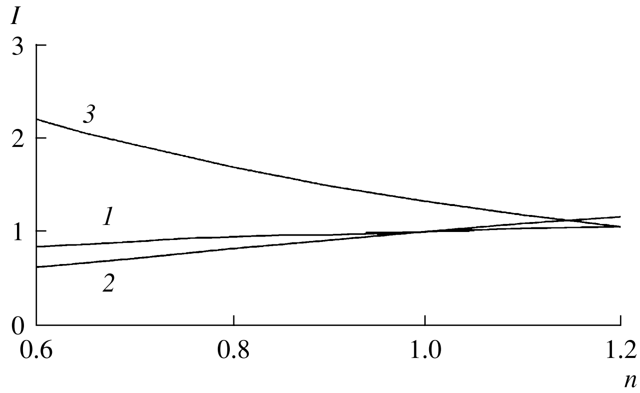

Fig. 2. (*1*) I_{St}, (2) I_{A-R}, and (*3*) I_P integrals as a function of the flow behavior index *n*.

is decreased by a factor of 1. 5 to 2 when the void fraction is as low as $\varphi = 0.05$, as compared to the analogous velocity for single bubbles. The largest decrease takes place for the Stokes flow with the Kuwabara and Happel boundary conditions. The relative velocity of the swarm of bubbles in the potential flow of a non-Newtonian liquid declines with increasing void fraction at a much slower rate. For example, when $\varphi = 0.05$, the decline in the bubble-swarm velocity as compared to the velocity of single bubbles is as low as 12 to 20%. Another distinctive feature in the motion of a swarm of bubbles through a non-Newtonian liquid is the size growth of bubbles (which keep their spherical shape) with increasing flow behavior index *n*. This growth takes place for all flows and boundary conditions. The latter can be attributed to a considerable increase in the effective viscosity of dilatant media (when $n > 1$).

The above regularities, which were found for the flow of a non-Newtonian liquid around a swarm of bubbles, are also valid for liquids with constant viscosity. As the void fraction in these liquids increases, the velocity of the flow around a swarm of bubbles likewise decreases [4, 5, 11], and the increase in the viscosity enhances the stability of their spherical shape [10, 12]. The fact that the velocity of a swarm of bubbles in the Stokes flow is slowed down to a relatively greater extent as compared to the Hadamard-Rybczynski and potential flows can be attributed to the state of the bubble surface. For the retarded surface of bubbles, the presence of bubbles in the gas–liquid system enhances the action of the flow resistance to their motion. The same action is produced by the Kuwabara and Happel boundary conditions because the requirement that the curl of the velocity or shear stresses produced by the

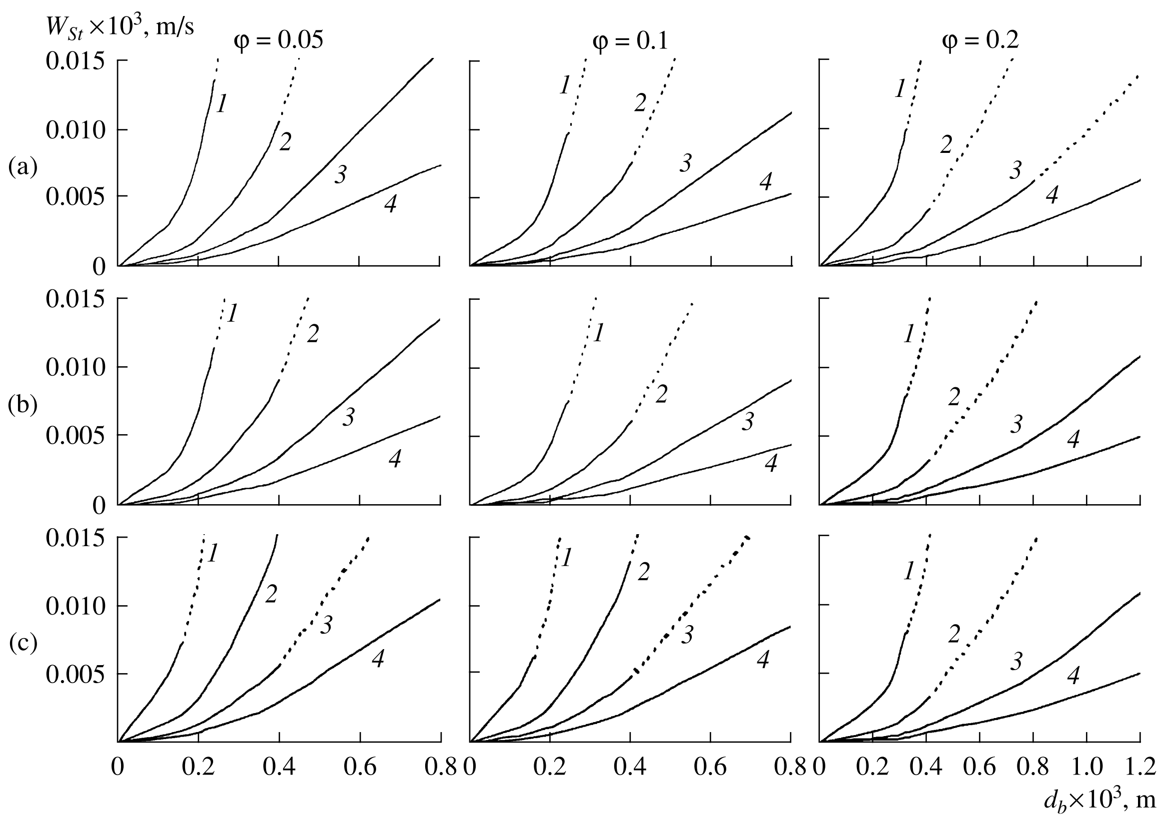

Fig. 3. Velocity of the Stokes flow of a non-Newtonian liquid around a swarm of bubbles for (a) Happel, (b) Kuwabara, and (c) Slobodov–Chepura boundary conditions at the cell surface at different void fractions φ in the gas–liquid system: n = (*1*) 0.6, (*2*) 0.8, (*3*) 1.0, (*4*) 1.2.

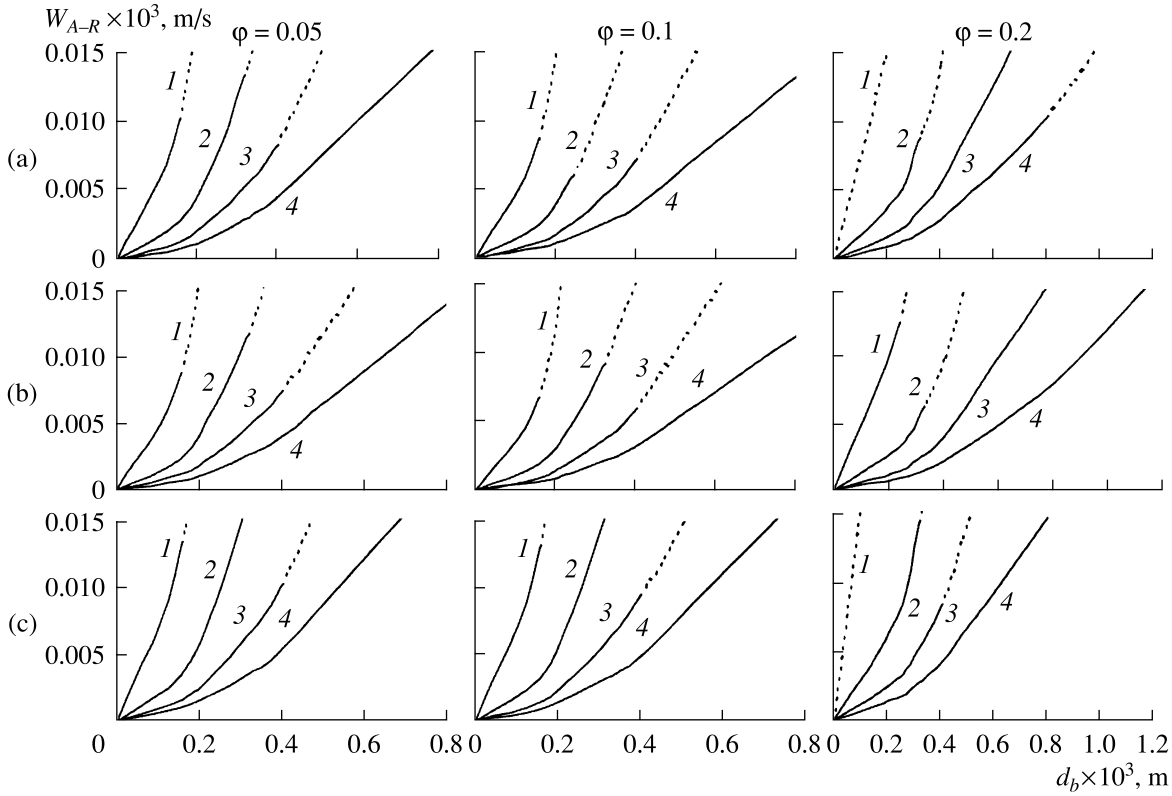

Fig. 4. Velocity of the Hadamard-Rybczynski flow of a non-Newtonian liquid around a swarm of bubbles for different boundary conditions at the cell surface at different void fractions φ in the gas–liquid system. Symbols as in Fig. 3.

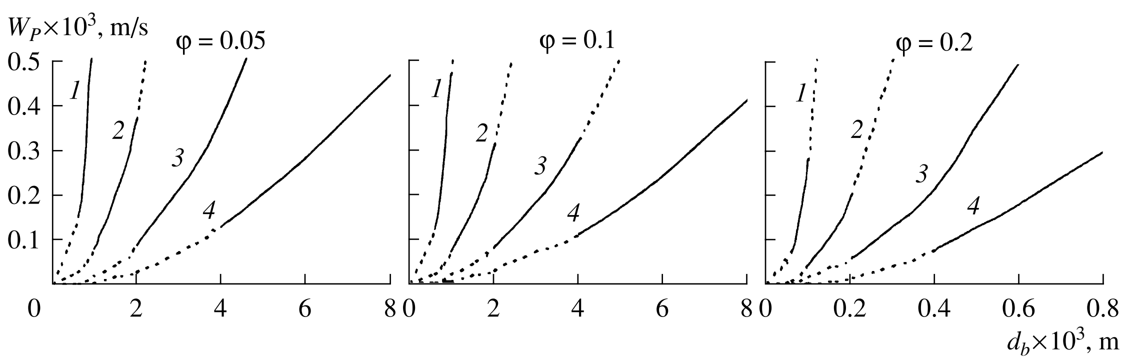

Fig. 5. Velocity of the potential flow of a non-Newtonian liquid around a swarm of bubbles at different void fractions φ in the gas–liquid system. Symbols as in Fig. 3.

velocity at the cell boundary should be equal to zero is more rigid than the requirement formulated for the derivative of the velocity curl in the Slobodov–Chepura condition.

NOTATION

a_i, b_i, a'_1, b'_3—coefficients involved in formulas (9) and (10);

d—diameter, m;

E—strain rate tensor, s^{-1};

g—acceleration of gravity, m/s^2;

K—consistency index for a non-Newtonian liquid, Pa s^n;

n—flow behavior index for the non-Newtonian liquid;

P—pressure, Pa;

P_0—static head of the liquid far away from the traveling bubble, Pa;

R—cell radius, m;

R_b—bubble radius, m;

r, θ, β—spherical coordinates attached to the center of the sphere in the flow;

S—bubble surface area, m^2;

t—time, s;

U—dissipated energy, J;

W—velocity of the swarm of bubbles relative to the continuous medium, m/s;

W_r, W_θ—radial and tangential components of the velocity of a continuous medium, m/s;

μ_b—viscosity of the gas contained in a bubble, Pa s;

ρ—density, kg/m^3;

σ—surface tension, N/m;

τ_{rr}, $\tau_{r\vartheta}$—normal and tangential components of the tensor of viscous stresses, Pa;

φ—void fraction in the gas–liquid system.

SUBSCRIPTS AND SUPERSCRIPTS

b—bubble;

ℓ—liquid;

St, *A–R*, *P*—Stokes, Hadamard-Rybczynski, and potential flows, respectively.

REFERENCES

1. Kutepov, A.M., Polyanin, A.D., Zapryanov, Z.D., et al., *Khimicheskaya gidrodinamika* (Chemical Fluid Dynamics), Moscow: Byuro Kvantum, 1996.
2. Kafarov, V.V. and Glebov, M.B., *Matematicheskoe modelirovanie osnovnykh protsessov khimicheskikh proizvodstv* (Mathematical Simulation of the Basic Unit Operations of the Chemical Industry), Moscow: Vysshaya shkola, 1991.
3. Brounshtein, B.I. and Fishbein, G.A., *Gidrodinamika, masso- i teploobmen v dispersnykh sistemakh* (Fluid Dynamics, Mass and Heat Transfer in Disperse Systems), Leningrad: Khimiya, 1977.
4. Gal-Or, B. and Waslo, S., Hydrodynamics of an Ensemble of Drops (or Bubbles) in the Presence or Absence of Surfactants, *Chem. Eng. Sci.*, 1968, vol. 23, p. 1431.
5. Rulev, N.N., Rise Velocity of a Swarm of Bubbles, *Kolloidn. Zh.*, 1977, vol. 39, no. 1, p. 80.
6. Cunningham, E., On the Velocity of Steady Fall of Spherical Particles through Fluid Medium, *Proc. R. Soc. London*, 1910, vol. 83A, no. 563, p. 357.
7. Happel, J., Viscous Flow in Multiparticle Systems: Slow Motion of Fluids Relative to Beds of Spherical Particles, *AIChE J.*, 1958, vol. 4, no. 2, p. 197.
8. Kuwabara, S., The Forces Experienced by Randomly Distributed Parallel Circular Cylinders or Spheres in a Viscous Flow at Small Reynolds Number, *J. Phys. Soc. Jpn.,* 1959, vol. 14, no. 4, p. 527.
9. Slobodov, E.B. and Chepura, I.V., On the Cell Model of Two-Phase Media, *Teor. Osn. Khim. Tekhnol.*, 1982, vol. 16, no. 3, p. 331.
10. Levich, V.G., *Fiziko-Khimicheskaya Gidrodinamika* (Physicochemical Fluid Dynamics), Moscow: Akad. Nauk SSSR, 1952.
11. Voinov, O.V. and Petrov, A.G., Flow of Bubbles through Liquids, in *Itogi nauki i tekhniki. Ser. “Mekhanika zhidkosti i gaza"* (Advances in Science and Technology, Ser.: Fluid Dynamics), Moscow: VINITI, 1976, vol. 10, p. 86.
12. Clift R., Grace J.R., Weber M.E. *Bubbles, Drops and Particles*, New York: Academic Press, 1978.