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Determining the Velocity of a Non-Newtonian Medium Flowing past Spherical Particles

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Abstract—The general hydrodynamic approach is used to formulate and solve the problem of determining the velocity field and pressure of a non-Newtonian medium flowing past a spherical particle. The rheological properties of the non-Newtonian medium are described in terms of a two-parameter power-law model. Expressions are obtained for the velocity of a sphere (in particular, a bubble) relative to the flow of the non-Newtonian liquid in different hydrodynamic pattern. The effects of rheological parameters on the flow velocity and on the limiting sizes of bubbles what retain spherical form, are analyzed.

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In this work, the spherical particle will be understood as a solid sphere, a liquid drop, or a gas bubble. From the hydro standpoint, these objects differ in their density relative to the continuous medium and in the state of the surface, which may be free (mobile) or blocked (quiescent). The former state is typical of drops and bubbles moving in a pure liquid containing no surfactant. The latter state is typical of solid particles, as well as drops and bubbles whose surface is blocked by an adsorbed surfactant.

The problem of determining the velocity of a continuous medium flowing past a spherical particle dates back to works by Stokes [1, 2], Hadamard [3], and Rybczynski [4]. In these works, they reported the solutions determining the velocity and pressure fields for the liquid flowing past a spherical particle and the velocity of this particle relative to the flowing continuous medium.

According to these solutions, at small Reynolds numbers (Re ≤ 1), the components of the velocity field of a non-Newtonian liquid flowing past a sphere and the pressure of the external medium on its surface has the following form in a spherical system of coordinates:

for the blocked surface of the sphere (Stokes flow pattern),

$$W_{r\,St} = W_{0\,St}\left(1 - \frac{3}{2}\frac{R_s}{r} + \frac{1}{2}\frac{R_s^3}{r^3}\right)\cos\theta,$$
$$W_{\theta\,St} = -W_{0\,St}\left(1 - \frac{3}{4}\frac{R_s}{r} - \frac{1}{4}\frac{R_s^3}{r^3}\right)\sin\theta, \quad (1)$$

$$P_{St} = P_0 - \frac{3}{2}\frac{\mu W_{0\,St} R_s}{r^2}\cos\theta$$

for the free surface of the sphere (Hadamard–Rybczynski flow pattern),

$$W_{r\,A-R} = W_{0\,A-R}\left(1 - \frac{R_s}{r}\right)\cos\theta,$$
$$W_{\theta\,A-R} = -W_{0\,A-R}\left(1 - \frac{R_s}{2r}\right)\sin\theta. \quad (2)$$

$$P_{A-R} = P_0 - \frac{\mu W_{0\,A-R} R_s}{r^2}\cos\theta$$

The velocities of the external medium relative to the sphere (bubble) for the corresponding hydrodynamic patterns (W_{0St} and W_{0A-R}) are determined from Eq. (1) and (2) subject to the boundary conditions far from the sphere ($r \longrightarrow \infty$) and on the sphere surface ($r = R_s$):

$$W_{0\,St} = \frac{2\Delta\rho g R_s^2}{9\mu}, \quad (3)$$

$$W_{0\,A-R} = \frac{\Delta\rho g R_s^2}{3\mu}. \quad (4)$$

At moderate values of the Reynolds number (100 ≤ Re ≤ 1000) in the potential flow, the components of the velocity field of the Newtonian liquid flowing past the sphere are given by the foolowing relationships, according to Levich's solution [5]:

$$W_{rP} = W_{0P}\left(1 - \frac{R_s^3}{r^3}\right)\cos\theta,$$
$$W_{\theta P} = W_{0P}\left(1 - \frac{R_s^3}{2r^3}\right)\sin\theta. \quad (5)$$

The velocity W_{0P} is

$$W_{0P} = \frac{\Delta\rho g R_s^2}{9\mu}. \quad (6)$$

As was mentioned above, all these results were obtained for a liquid with a constant viscosity μ. However, there is a rather large class of non-Newtonian liquids, whose effective viscosity μ_{ef} is a variable depending on the flow velocity (shear rate). The rheological behavior of such liquids are widely described in terms of a two-parameter power-law model, in which μ_{ef} is given by the following equation [6, 7]:

$$\mu_{ef} = KE^{n-1}, \quad (7)$$

where E is the strain rate intensity tensor, which is a multidimensional analogue of the shear rate.

The purpose of this work is to obtain analytical expressions for the velocity field components and pressure for different hydrodynamic patterns of non-Newtonian liquid flow past a sphere and to determine the velocity of this sphere relative to the flowing continuous medium.

We will use the approach applied by V.G. Levich to a similar problem for a Newtonian liquid. In this approach, the functions $W_r(r, \theta)$, $W_\theta(r, \theta)$, and $P(r, \theta)$ are represented as power series. The general expressions for the sought velocities and pressure of external medium will then appear as [5]

$$W_r = \left(\frac{b_1}{r^3} + \frac{b^2}{r} + b_3 + a_1 r^2\right)\cos\theta,$$
$$W_\theta = \left(\frac{b_1}{2r^3} - \frac{b^2}{2r} - b_3 - 2a_1 r^2\right)\sin\theta, \quad (8)$$
$$P_l - P_{0l} = KE^{n-1}\left(\frac{b_2}{r^2} + 10a_1 r\right)\cos\theta.$$

In order to solve the problem thus formulated, we will write similar expressions for the velocities and pressure inside the sphere, designating the corresponding components W'_r, W'_θ, and P_s. Since the velocity and pressure values in the center of the sphere (at $r \longrightarrow 0$) are finite, the terms containing the variable r in the denominator should be omitted. In view of this, the above expressions will take the form of

$$W'_r = (b'_3 + a'_1 r^2)\cos\theta,$$
$$W'_\theta = -(b'_3 + 2a'_1 r^2)\sin\theta. \quad (9)$$
$$P_s - P_{0s} = 10a'_1 r\mu_s\cos\theta.$$

In this formulation, the problem reduces to determining the unknown coefficients a_1, b_1, b_2 and b_3 for particular interfacial conditions and a particular state of the surface of the spherical particle. Taking into account the flow pattern examined, we will use the spherical coordinate system associated with the center of the sphere. The sphere will be considered to *be motionless*, and the external medium will be considered to flow past the sphere in the opposite direction with a velocity $-W_0$ (W_0 is the velocity of the sphere itself).

The boundary conditions (values of components of the velocity field and pressure) will be defined on the sphere surface, which is the interface ($r = R_s$), and far from the sphere ($r \longrightarrow \infty$). Far from the moving sphere (bauble, drop, or solid particle), the distribution of velocities and pressure in the liquid medium (either Netwonian and non-Newtonian) has the following form [5–7]:

$$W_{r|r\to\infty} = W_0\cos\theta, \quad W_{\theta|r\to\infty} = -W_0\sin\theta. \quad (10)$$

At the interface, the radial component of the velocity of the external medium flowing past the sphere (W_r) and that of velocity of the internal flow circulating in the sphere (W'_r) are zero, while the tangential velocities W_θ and W'_θ, and the components of the viscous stress tensor, both normal (τ_{rr}, τ'_{rr}) and tangential ($\tau_{r\theta}$, $\tau'_{r\theta}$), remain continuous. In view of expression (7), the boundary conditions for the velocity components will appear as

$$W_{r|r=R_s} = 0, \quad (11)$$

$$W'_{r|r=R_s} = 0, \quad (12)$$

$$W_{\theta|r=R_s} = W'_{\theta|r=R_s}, \quad (13)$$

$$P_s - P_{l|r=R_s} = 2\mu_s\frac{\partial W'_r}{\partial r} - 2KE^{n-1}\frac{\partial W_r}{\partial r} + (\rho_l - \rho_s)gR_s\cos\theta_{|r=R_s}, \quad (14)$$

$$KE^{n-1}\left(\frac{1}{r}\frac{\partial W_r}{\partial\theta} + \frac{\partial W_\theta}{\partial r} - \frac{W_\theta}{r}\right)_{|r=R_s} = \mu_s\left(\frac{1}{r}\frac{\partial W'_r}{\partial\theta} + \frac{\partial W'_\theta}{\partial r} - \frac{W'_\theta}{r}\right)_{|r=R_s}. \quad (15)$$

In the right-hand side of Eq. (14), the term $\Delta\rho g R_s\cos\theta$ is added to the pressure of the external liquid. This term accounts for the growth of the external hydrostatic pressure along the height of the sphere (at $\theta = 0$ (upper pole), $\cos\theta = 1$ and P_l is lower than at $\cos\theta = \pi$ (lower pole), where P_l is higher). The strain rate intensity tensor of a non-Newtonian liquid on the sphere surface ($r = R_s$) in terms of the velocity components is expressed as follows [7]:

$$E = \left[2\left(\frac{\partial W_r}{\partial r}\right)^2 + 2\left(\frac{1}{r}\frac{\partial W_\theta}{\partial \theta}\right)^2 + 2\left(\frac{W_\theta}{r}\cot\theta\right)^2 + \left(\frac{\partial W_\theta}{\partial r} - \frac{W_\theta}{r} + \frac{1}{r}\frac{\partial W_r}{\partial \theta}\right)^2\right]^{\frac{1}{2}}. \quad (16)$$

Here, it is taken into account that $W_\varphi \equiv 0$ because we assume that flow around the sphere is symmetrical with respect to the circular coordinate and $W_r = 0$, according to the boundary condition (11).

To satisfy the boundary conditions (10), the constant a_1 in Eq. (8) should be taken to be zero (the flow velocity far from the sphere must be finite). It will the follow from the same boundary conditions that $b_3 = W_0$, where W_0) is the velocity of a single sphere [5]. The other unknown coefficients can be determined by substituting the expressions for the velocity field components and pressure (Eqs. (8) and (9)) into the boundary condition equations (11)–(15) and into the related equation for the strain rate intensity tensor (16). The resulting algebraic equations are nonlinear. They can be solved explicitly for the sought coefficients b_1–b_3 only in two limiting cases: $\mu_s \longrightarrow \infty$ (the sphere in the flowing medium is a solid particle or a bubble whose surface is blocked) and $\mu_s \longrightarrow \infty$ (the sphere in the flowing medium is a bubble or a drop with a free surface). However, these cases correspond to the most important hydrodynamic patterns, namely, the Stokes and Hadamard–Rybczynski patterns.

For the *Stokes pattern*, the expression for the strain rate intensity (Eq. (16)) will appear as $E = \left(\frac{\Delta\rho g \sin\theta R_s}{3K}\right)^{\frac{1}{n}}$ (hereafter, $\Delta\rho = \rho_l - \rho_s$). The unknown coefficients in Eq. (8) are

$$b_1 = \frac{\Delta\rho g}{9K}R_s^5 E^{1-n}, \quad b_2 = -\frac{\Delta\rho g}{3K}R_s^3 E^{1-n},$$
$$b_3 = \frac{2\Delta\rho g}{3K}R_s^2 E^{1-n}.$$

The quantities appearing in the expression for E are constant, except θ (angular coordinate); that is, the strain rate intensity varies over the surface of the sphere. This feature differentiates E from the one-dimensional shear rate γ. In the subsequent calculations, we will use E^{1-n} averaged out over the surface S of the spherical particle:

$$E_{St}^{1-n} = \left(\frac{\Delta\rho g R_s}{3K}\right)^{\frac{1-n}{n}} I_{St}, \quad (17)$$

where

$$I_{St} = \frac{1}{4\pi R_s^2}\iint_S \sin^{\frac{1-n}{n}}\theta\, dS = \frac{1}{2}\int_0^\pi \sin^{\frac{1}{n}}\theta\, d\theta. \quad (18)$$

In view of this, the sought coefficients are equal to

$$b_1 = \left(\frac{\Delta\rho g}{3^{n+1}K}\right)^{\frac{1}{n}} I_{St} R_s^{\frac{4n+1}{n}}, \quad b_2 = -\left(\frac{\Delta\rho g}{3K}\right)^{\frac{1}{n}} I_{St} R_s^{\frac{2n+1}{n}},$$
$$b_3 = 2\left(\frac{\Delta\rho g}{3^{n+1}K}\right)^{\frac{1}{n}} I_{St} R_s^{\frac{n+1}{n}}.$$

By substituting these values into Eq. (8), we obtain expressions for velocity field components and pressure for a non-Newtonian liquid flowing past a single sphere:

$$W_{r\,St} = \left(\frac{2^n \Delta\rho g R_s^{n+1}}{3^{n+1}K}\right)^{\frac{1}{n}} I_{St}\left(1 - \frac{3}{2}\frac{R_s}{r} + \frac{1}{2}\frac{R_s^3}{r^3}\right)\cos\theta,$$
$$W_{\theta\,St} = -\left(\frac{2^n \Delta\rho g R_s^{n+1}}{3^{n+1}K}\right)^{\frac{1}{n}} I_{St}\left(1 - \frac{3}{4}\frac{R_s}{r} - \frac{1}{4}\frac{R_s^3}{r^3}\right)\sin\theta, \quad (19)$$
$$P_{l\,St} = P_{0l} - \frac{\Delta\rho g R_s^3}{3r^2}\cos\theta.$$

The velocity of a single sphere relative to the external flow of a non-Newtonian liquid, W_{0St}, can be determined as the value of the coefficient b_3 (see above):

$$W_{0\,St} = \left(\frac{2^n \Delta\rho g R_s^{n+1}}{3^{n+1}K}\right)^{\frac{1}{n}} I_{St}. \quad (20)$$

At $n = 1$ (Newtonian liquid), $K = \mu$, $I_{St} = 1$, and formula (20) reduces to the familiar Stokes formula (3) and Eq. (19) reduces to Eq. (1) for the velocity field components and pressure in the case of a Stokes flow of a Newtonian liquid past a single particle.

For the *Hadamard–Rybczynski pattern*, similar calculations lead to the following expression for the strain rate intensity: $E = \left(\frac{\Delta\rho g R_s \cos\theta}{3^{1/2}K}\right)^{\frac{1}{n}}$, . Hence, we obtain the sphere surface–averaged value of E:

$$E_{A-R}^{1-n} = \left(\frac{\Delta\rho g R_s}{3^{1/2}K}\right)^{\frac{1-n}{n}} I_{A-R}, \qquad (21)$$

where

$$I_{A-R} = \frac{1}{4\pi R_s^2}\iint_S \cos^{\frac{1-n}{n}}\theta dS = \frac{1}{2}\int_0^{\pi} \cos^{\frac{1-n}{n}}\theta \sin\theta d\theta. \qquad (22)$$

The coefficients b_1, b_2, and b_3 will be equal to

$$b_1 = 0, \quad b_2 = -\left(\frac{\Delta\rho g}{3^{\frac{n+1}{2}}K}\right)^{\frac{1}{n}} I_{A-R} R_s^{\frac{2n+1}{n}},$$

$$b_3 = \left(\frac{\Delta\rho g}{3^{\frac{n+1}{2}}K}\right)^{\frac{1}{n}} I_{A-R} R_s^{\frac{n+1}{n}}.$$

The sought expressions for the velocity field components and pressure will appear as

$$W_{r\,A-R} = \left(\frac{\Delta\rho g R_s^{n+1}}{3^{\frac{n+1}{2}}K}\right)^{\frac{1}{n}} I_{A-R}\left(1 - \frac{R_s}{r}\right)\cos\theta,$$

$$W_{\theta\,A-R} = -\left(\frac{\Delta\rho g R_s^{n+1}}{3^{\frac{n+1}{2}}K}\right)^{\frac{1}{n}} I_{A-R}\left(1 - \frac{R_s}{2r}\right)\sin\theta, \qquad (23)$$

$$P_{l\,A-R} = P_{0l} - \frac{\Delta\rho g R_s^3}{3r^2}\cos\theta.$$

The expressions for pressure for the Stokes and Hadamard–Rybczynski flow patterns turned out to be identical because the pressure is determined by the normal component of the viscous stress tensor, τ_{rr}, which is independent of the state of the surface of the sphere in the flowing medium. Mathematically, this is manifested as the fact that the multipliers KE^{n-1} and b_2 in the general expression for pressure (Eq. (8)) are mutually canceled because the latter contains E to the power (1 – n). The velocity of a single sphere in this flow pattern determined in terms of the coefficient b_3, as in the Stokes flow patterns:

$$W_{0\,A-R} = \left(\frac{\Delta\rho g R_s^{n+1}}{3^{\frac{n+1}{2}}K}\right)^{\frac{1}{n}} I_{A-R}. \qquad (24)$$

Just as in the Stokes pattern, at $n = 1$ $I_{A-R} = 1$ and formula (24) will be identical to Eq. (4), which describes the velocity (23) will be identical to the components of the velocity field and pressure for the Hadamard–Rybczynski flow of a Newtonian medium past a bubble (Eq. (2)).

For the *potential pattern* of the motion of the sphere, the flow of a liquid (either Newtonian or non-Newtonian) around it can be viewed as being ideal; that is, it is possible to neglect the energy dissipation arising from to the viscous friction in the bulk of the continuous medium. As distinct from the above analysis of the flow past spherical particles in the Stokes and Hadamard–Rybczynski pattern, in which no constraints were imposed on the rheological properties of the continuous medium and the interfacial physical effects were taken into account as the ratio of the viscosities of the sphere and the continuous medium, the analysis of this hydrodynamic pattern will deal with a more concretely defined surface of the sphere: it will be assumed that the surface of the sphere is free (mobile). The manifestation of the viscous properties of the liquid in this case is localized in a boundary layer that is thin as compared to the size of the sphere. This assumption makes it possible to introduce and use the ideal liquid model, whose flow outside the boundary layer is irrotational ($\mathrm{rot}_\varphi \vec{W} = 0$). This approach, which was also suggested by Levich [5], reduces solving the Navier–Stokes nonlinear equations to solving the Laplace linear equation, from which one can derive the velocity fields components for the ideal liquid flowing past the sphere for the potential pattern:

$$W_{r\,P} = W_{0\,P}\left(1 - \frac{R_s^3}{r^3}\right)\cos\theta,$$

$$W_{\theta\,P} = W_{0\,P}\left(1 - \frac{R_s^3}{2r^3}\right)\sin\theta. \qquad (25)$$

Here, W_{0P} is the velocity of the liquid far from the sphere.

The velocity distribution near the sphere surface (at $r \approx R_s$), where the viscous forces show themselves, will be determined by introducing the new variable $y = R_s - r$ and expanding expression (25) in a series in y/R_s powers [5]:

$$W_{r\,P|\,r\approx R_s} = -3W_{0\,P}\frac{y}{R_s}\cos\theta,$$

$$W_{\theta\,P|\,r\approx R_s} = \frac{3}{2}W_{0\,P}\left(1 - \frac{y}{R_s}\right)\sin\theta. \qquad (26)$$

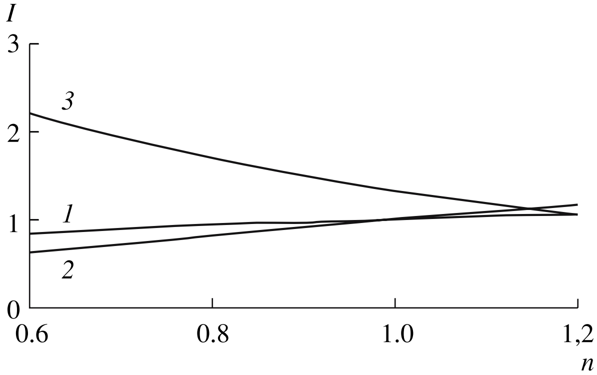

Fig. 1. Integrals (*1*) I_{St}, (*2*) I_{A-R}, and (*3*) I_P versus the rheological parameter *n*.

The amount of energy dissipated in the boundary layer (*dU*/*dt*), where the viscous forces are manifested, is equal to

$$-\frac{dU}{dt_{|r\approx R_s}} = KE^{n-1}\iint_S \frac{\partial \vec{W}^2_{|r\approx R_s}}{\partial \vec{n}} dS. \quad (27)$$

Here, $\frac{\partial \vec{W}^2_{|r\approx R_s}}{\partial \vec{n}} = \frac{\partial}{\partial r_{|r\approx R_s}}(W^2_{rP} + W^2_{\theta P})$ is the partial derivative of the velocity vector of th external flow near the sphere surface and $dS = R_s^2 \sin\theta d\theta d\varphi$ is the surface element.

The value of *E* in Eq. (27) is derived from expression (16) using the velocity field components defined by formulas (26). These calculations yield

$$-\frac{dU}{dt_{|r\approx R_s}} = 3^{\frac{3n+1}{2}}\pi K W_{0P}^{n+1} R_s^{2-n} I_P, \quad (28)$$

where

$$I_P = \int_0^\pi \cos^{n-1}\theta \sin^3\theta d\theta \text{ (at } n = 1\ I_P = 4/3). \quad (29)$$

The total dissipative force (drag) F_d with which the non-Newtonian liquid flow acts on the sphere is expressed as follows [5]:

$$F_d = -\frac{1}{2}\frac{\partial}{\partial W_{0P}}\left(\frac{dU}{dt}\right) = \frac{3^{\frac{3n+1}{2}}}{2}(n+1)\pi K W_{0P}^n R_s^{2-n} I_P. \quad (30)$$

By equating this drag force to the Archimedes lift acting on the sphere, we obtain the velocity of the sphere relative to the non-Newtonian liquid for the potential flow pattern:

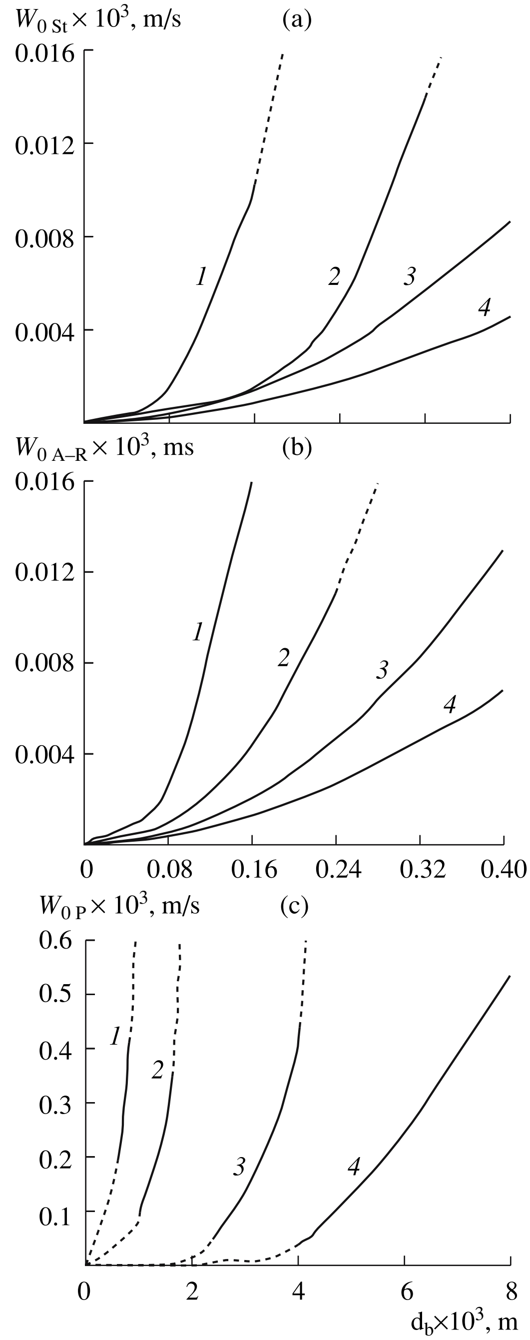

Fig. 2. Velocity of the non-Newtonian liquid flow past a single sphere (bubble) versus the rheological parameter *n* for the (a) Stokes, (b) Hadamard–Rybczynski, and (c) potential flow patterns: *n* = (*1*) 0.6, (*2*) 0.8, (*3*) 1.0, and (*4*) 1.2. The dashed lines are the portions of the curves in which the bubble size is beyond the limits of the corresponding hydrodynamic patterns.

$$W_{0P} = \left[\frac{8\Delta\rho g R_s^{n+1}}{27^{\frac{n+1}{2}}(n+1)I_P K}\right]^{\frac{1}{n}}. \quad (31)$$

At $n = 1$, this expression reduces to the Levich formula (6) for the velocity of a single sphere (bubble) in a Newtonian liquid in the potential flow pattern. Integrals (18), (22), and (29), which depend on the rheological parameter n, are calculated numerically (Fig. 1).

Figure 2 plots the velocity of a non-Newtonian liquid flowing past a single sphere (bubble) calculated using formulas (20), (24), and (31) for the corresponding hydrodynamic patterns. In these calculations, we accepted the following values of the constants: $K = 0.01$ Pa s^n, c^n, $\Delta\rho = 998$ kg/m^3, and $g = 9.81$ m/s^2. The bubble size limits within which the given hydrodynamic pattern of rising takes place were defined by recurrent relationships (W_0 is the velocity of the bubble in the given pattern):

for the Stokes and Hadamard–Rybczynski pattern,

$$d_b \le \left(\frac{K}{\Delta\rho W_0^{2-n}}\right)^{\frac{1}{n}};$$

for the potential flow pattern, $\left(\frac{100K}{\Delta\rho W_0^{2-n}}\right)^{\frac{1}{n}} \le d_b \le \left(\frac{1000K}{\Delta\rho W_0^{2-n}}\right)^{\frac{1}{n}}$.

An analysis of the curves shown in Fig. 2 demonstrates that, for all hydrodynamic regimes, the bubble velocity relative to the non-Newtonian medium decreases dramatically as the flow index n increases. This effect can be explained by the rheological features of the non-Newtonian liquids: for pseudoplastics ($n < 1$), the effective viscosity in the vicinity of a bubble decreases as the shear rate increases; for dilatant liquids ($n > 1$), it increases with an increasing shear rate. At a constant external force (lift) acting on the bubble, this leads to an increase (in the former case) or a decrease (in the latter case) in the equilibrium liquid velocity relative to the velocity of the Newtonian liquid.

It is also clear from the curves in Fig. 2 that the Stokes and Hadamard–Rybczynski patterns at a given consistency index K of the non-Newtonian liquid persist for bubbles up to 0.6–0.7 mm in diameter. The upper limit of this size interval is typical of dilatant liquids ($n > 1$). For pseudoplastic media, the bubble sizes at which these hydrodynamic patterns take place are much smaller (on the order of 0.2–0.3 mm) and these sizes decrease rapidly as the parameter n and, accordingly, the effective viscosity of the non-Newtonian medium decrease. The difference between the pseudoplastic and dilatant liquids is still clearer in the potential flow pattern: at a fixed K value (0.01 Pa s^n), the bubble size interval corresponding to the potential pattern of rising is 0.6–2.0 mm for the former and 4–7 mm for the latter.

The question of whether the bubbles of these sizes retain their spherical shape can be answered by the following estimation procedure. From the physical standpoint, the sphericity of the bubble persists when the surface tension forces, which act to reduce the interfacial area (i.e., to give the shape of a sphere to a bubble in a liquid), exceed the hydrodynamic pressure of the liquid flowing pas the bubble, which acts to distort the bubble surface. Mathematically, this condition for a liquid with a constant (Newtonian) viscosity is expressed as follows [5]:

$$R_b \le \left(\frac{324\sigma\mu^2}{\rho_l^3 g^2}\right)^{\frac{1}{5}}. \quad (32)$$

For a dilatant non-Newtonian medium flowing past a bubble with a velocity belonging to the calculated interval, the effective viscosity will be of the order of $(2–3) \times 10^{-2}$ Pa s^n. At a surface tension of $\sigma = (40–50) \times 10^{-3}$ N/m, this viscosity value leads to a critical bubble size (diameter) of 4.5–6 mm for sphericity to be retained. This result correlates with the limiting size that we calculated for bubbles rising in the potential flow pattern.

In conclusion, note that expressions (20), (24), (31) for the velocity of the non-Newtonian liquid flow past a sphere are identical to the expressions for the motion of spherical particles in these media, but only in the case of solid particles and drops. The velocity of gas bubbles moving (rising) in a liquid will change as the bubble rises because of the variation of the bubble size. This explained by the fact that the hydrostatic pressure P_{0l} depends on the depth at which the bubble is. The gas pressure in the bubble, P_{0b}, is higher than the pressure of the surrounding liquid by the value of the capillary pressure $2\sigma/R_b$, and the value of $P_{0b} - P_{0l}$ grows as the bubble rises. Dynamic equilibrium in this process is due to the increasing size of the rising bubble and the corresponding decrease of the capillary pressure in it.

NOTATION

a_1, a_1', b_1, b_2, b_3'—coefficients in Eqs. (8) and (9);

d—diameter, m;

E—strain rate intensity tensor, s^{-1};

F_d—total dissipative force (drag), N;

g—acceleration of gravity, m/s^2;

I—numerical value of the integrals give by formulas (18), (22), and (29) for the hydrodynamic patterns examined;

K—rheological parameter, specifically, the consistency of the non-Newtonian medium, Pa s^n;

n—rheological parameter, specifically, the flow index of the non-Newtonian medium;

P—pressure, Pa;

P_0—hydrostatic pressure of the liquid far from the moving particle, Pa;

R—radius of the spherical particle, m;

t—time, s;

U—dissipated energy, J;

W_0—velocity of the spherical particle relative to the continuous medium, m/s;

W_r, W_θ—radial and tangential components of the velocity of the continuous medium, m/s;

r, θ, φ—spherical coordinates associated with the center of the particle in the flowing medium;

μ—viscosity of the Newtonian medium, Pa s;

μ_{ef}—effective viscosity of the non-Newtonian medium, Pa s;

ρ—density, kg/m^3;

σ—surface tension, N/m.

SUBSCRIPTS AND SUPERSCRIPTS

b—bubble;

l—liquid;

s—spherical particle;

St, A–R, P—Stokes, Hadamard–Rybczynski, and potential hydrodynamic flow patterns, respectively.

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