Theoretical Foundations of Chemical Engineering, Vol. 39, No. 5, 2005, pp. 529–536.
Translated from Teoreticheskie Osnovy Khimicheskoi Tekhnologii, Vol. 39, No. 5, 2005, pp. 560–567.

Calculation of the Efficiency of Inertialess Flotation of Disklike Particles

O. M. Sokovnin and N. V. Zagoskina
Vyatka State Technical University, ul. Moskovskaya 36, Kirov, 610000 Russia
e-mail: osokovnin@mail.ru
Received May 11, 2004

Abstract—The hydrodynamics of the flotation entrainment of disklike particles by a rising bubble is considered. Using stochastic methods for specifying the orientation of particles with respect to the bubble surface, analytical expressions for the particle entrainment efficiency for the potential and Stokes bubble rise patterns are obtained.

In recent decades, inertialess flotation theory has been actively developed [1, 2], which is related to the demand for energy-saving technologies for separating liquid heterogeneous systems. The main step of an elementary act of inertialess flotation is the collision of a particle with a rising bubble with the subsequent entrainment of the particle by the bubble. The efficiency of this process is determined by the ratio between the sizes of the particle and the rising bubble and by the bubble rise pattern. The expressions for the efficiency of the flotation entrainment of the isometric (spherical) particles are known [3]:

$$E_{\text{pot}} = 3\frac{r_{\text{p}}}{R_{\text{b}}}, \tag{1}$$

$$E_{\text{St}} = \frac{3}{2}\left(\frac{r_{\text{p}}}{R_{\text{b}}}\right)^2. \tag{2}$$

For anisometric particles, we proposed a mathematical model for calculating the flotation entrainment efficiency using a stochastic approach [4, 5]. In terms of this model, we consider the entrainment of disklike particles by a rising bubble.

We suppose that a disklike particle is flat (its radius r_{p} is much larger than its transverse size) and, moreover, $R_{\text{b}} \gg r_{\text{p}}$. We also assume that the particle density ρ_{p} is close to the density ρ_{l} of the flotation medium. Under these conditions, which are characteristic of inertialess flotation, one can suppose that the particle moving along streamlines is oriented randomly with respect to the bubble. The set of possible equiprobable orientations of the particle forms a sphere of radius r_{p} around the particle.

To solve the problem, we consider the relative motion of the bubble and the particle by introducing orthogonal coordinate systems, namely, a Cartesian coordinate system (x, y, z) and a spherical coordinate system (r, α, β), whose origin (the point O) is at the center of the bubble (the large sphere). The Ox axis of the Cartesian coordinates coincides with the segment OO' connecting the centers of the bubble and the particle (i.e., the coordinate system (x, y, z) rotates about the center of the bubble). The meridional spherical coordinate α is defined in the plane xOz and takes values from $-\pi/2$ to $\pi/2$. The origin of the circumferential spherical coordinate β is in the plane zoy, and $\beta \in [0, 2\pi]$ (Fig. 1).

Let us represent the spherical coordinates in terms of the Cartesian coordinates. The projection of a certain arbitrary point A (r, α, β) on the axis x is the point C (Fig. 1) with the coordinates

$$\begin{cases} x = r\cos\beta\cos\alpha \\ y = r\sin\beta \\ z = r\cos\beta\sin\alpha. \end{cases} \tag{3}$$

As previously [4, 5], for particles moving along streamlines, we define the entrainment efficiency as

$$E = \frac{2}{R_{\text{b}}^2}\int_0^{b_c} p(s)s\,ds. \tag{4}$$

Here, the quantity b_c is determined by the ratio between the sizes of the bubble and the particle and also by the bubble rise pattern [1, 3].

Earlier, in solving the problem of entrainment of a rodlike particle by a bubble, the probability $p(s)$ was defined as the ratio of the double area F of the region that is cut from the small sphere of radius l to the surface area of the bubble during their relative motion [4, 5]:

$$p(s) = \frac{2F}{4\pi l^2},$$

where l is half of the rodlike particle length.

(The double value of the area F is taken because the rodlike particle can touch the bubble with each of its two ends.)

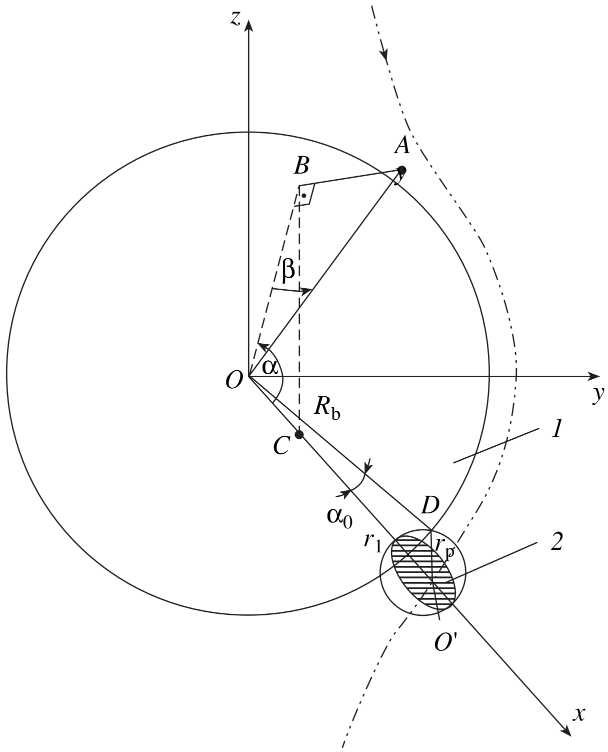

Fig. 1. Relative position of bubble *1* and disklike particle *2* in the neighborhood of the bubble; (x, y, z) and (r, α, β) are the orthogonal Cartesian and spherical coordinate systems, respectively.

In the case we consider now, the disklike particle is flat and, when it touches the bubble, a part of the volume V, rather than a part of the surface area, is cut from the small sphere of radius r_p; i.e.,

$$p(s) = \frac{2V}{\frac{4}{3}\pi r_p^3} = \frac{3V}{2\pi r_p^3}. \quad (5)$$

(The double value of the volume is taken because the disk may touch the bubble with each of the halves of the outer arc: when on the nearest streamline, where the distance $O'O$ between the centers of the bubble and the particle at $\alpha = 0$ (Fig. 1) is R_b, i.e., the probability of touching is unity, the bubble cuts out only half of the volume of the small sphere.)

Thus, solving the problem is reduced to finding the volume V of the small sphere that is cut out by the bubble surface. In the accepted coordinate system (r, α, β), the expression for this volume has the form

$$V = 4\int_0^{\alpha_0} d\alpha \int_0^{\beta_0} \cos\beta d\beta \int_{r_1}^{R_p} r^2 dr. \quad (6)$$

Integration with respect to the variables α and β ($\alpha \in [-\alpha_0, \alpha_0]$ and $\beta \in [-\beta_0, \beta_0]$) is performed twice over the halves of the symmetric regions $[0, \alpha_0]$ and $[0, \beta_0]$; therefore, with allowance for the symmetry, the factor 4 appears in front of the integral in expression (6).

The limits of integration in the multiple integral are found as follows.

r_1 characterizes the inner boundary of the region cut from the small sphere (Fig. 1) and is determined from the equation of the small sphere in the Cartesian coordinate system:

$$(x - r_\theta)^2 + y^2 + z^2 = r_p^2. \quad (7)$$

Substitution of expressions (3) for x, y, and z into Eq. (7) with the subsequent transformations yields

$$r_1 = r_\theta \cos\beta\cos\alpha - \sqrt{r_\theta^2\cos^2\beta\cos^2\alpha + r_p^2 - r_\theta^2}. \quad (8)$$

α_0 is the maximal meridional angle formed by the intersection of the small and large spheres in the plane zOx (Fig. 1); α_0 is determined from the law of cosines for the triangle ODO':

$$\alpha_0 = \arccos\frac{r_\theta^2 + R_b^2 - r_p^2}{2R_b r_\theta}. \quad (9)$$

The maximal circumferential angle β_0 formed by the intersection of the spheres depends on the current coordinate α ($\alpha \in [0, \alpha_0]$). The section of the small sphere with a certain plane $\pi(\alpha)$ constructed on the axis Oy is a circle of radius $O''E = \sqrt{r_p^2 - r_\theta^2\sin^2\alpha}$ (Fig. 2a). The sought central angle $\beta_0(\alpha)$ formed by the intersection of this circle with the large sphere in the plane $\pi(\alpha)$ is also found from the law of cosines for the triangle $OO''E$:

$$OO'' = r_\theta\cos\alpha, \quad OE = R_b,$$

$$O''E = \sqrt{r_p^2 - r_\theta^2\sin^2\alpha} \quad \text{(Fig. 2)},$$

$$\beta_0(\alpha) = \arccos\frac{r_\theta^2 + R_b^2 - r_p^2}{2R_b r_\theta\cos\alpha}. \quad (10)$$

Integrating expression (6) with respect to r within the mentioned limits, estimating the order of magnitude of the obtained terms with allowance for specific features of the problem under consideration ($r_\theta \sim R_b \gg r_p$) and expressions (9) and (10) (whence it follows that the angles α_0 and β_0 are small), and performing certain transformations, we obtain

$$V = \frac{4}{3}\int_0^{\alpha_0} d\alpha$$
$$\times \int_0^{\beta_0(\alpha)} (R_b^3 - 4r_\theta^3\cos^3\beta\cos^3\alpha + 3r_\theta^3\cos\beta\cos\alpha \quad (11)$$
$$+ 3r_\theta^2\cos^2\beta\cos^2\alpha\sqrt{r_\theta^2\cos^2\beta\cos^2\alpha + r_p^2 - r_\theta^2})\cos\beta d\beta.$$

Integration with respect to β is performed with allowance for the fact that the upper limit β_0 is a function of α (see expression (10)). For this integration, the functions whose argument is β ($\beta \in [0, \beta_0] \le 0.1$) are expanded into Taylor series, which are next truncated to the second-order terms. Then, $\beta_0(\alpha)$ and functions involving β_0 take the form

$$\cos\beta_0(\alpha) = 1 - \frac{\alpha_0^2 - \alpha^2}{2}, \quad (12)$$

$$\beta_0(\alpha) \approx \sin\beta_0(\alpha) = \sqrt{\alpha_0^2 - \alpha^2} \approx \alpha_0\left(1 - \frac{\alpha^2}{2\alpha_0}\right). \quad (13)$$

Taking into account these expressions, the subsequent similar series expansion of the functions of the argument α ($\cos\alpha$, $\cos^2\alpha$, etc.), and expression (9) for α_0, we make some transformations and calculate integral (11), thus obtaining the sought expression for the elementary volume V cut from the small sphere with the bubble surface:

$$V = \frac{10}{9}\frac{r_p^2}{R_b r_\theta}(R_b^3 - r_\theta^3) + \frac{4}{3}\pi\frac{r_p^3 r_\theta^2}{\sqrt{R_b r_\theta}}\left(\frac{1}{r_\theta} - \frac{1}{3R_b}\right).$$

Here, the probability $p(s)$ in accordance with expression (5) is

$$p(s) = \frac{5(R_b^3 - r_p^3)}{3\pi r_p R_b r_\theta} + 2\frac{r_\theta^2}{\sqrt{R_b r_\theta}}\left(\frac{1}{r_\theta} - \frac{1}{3R_b}\right). \quad (14)$$

This expression quantitatively determines the probability that the surface of the bubble of the radius R_b is touched by the randomly oriented disklike particle of the radius r_p that is located at the distance r_θ ($R_b \le r_\theta \le R_b + r_p$).

To find the efficiency of entrainment of the disklike particle moving past the rising bubble along a streamline, we consider their relative motion in the spherical coordinate system (r, θ, φ). The small sphere formed by the set of random orientations of the disklike particle with respect to the bubble moves along a certain streamline and touches the bubble surface at the point with the coordinates $r = R_b + r_p$ and $\theta = \theta_0$. The angle θ_0 value depends on the distance s from the given streamline to the axis of motion of the rising bubble (Fig. 3). Thus, in determining the entrainment efficiency E from expression (4), the integration of $p(s)$ is initially performed with respect to the angular coordinate θ from θ_0 to $(\pi - \theta_0)$ or, with allowance for the symmetry of the considered flow about the equatorial plane of the bubble, twice from θ_0 to $\pi/2$, and then with respect to the radial coordinate from 0 (the axis of motion of the bubble) to b_c (the limiting streamline on which the particle still can touch the bubble surface). With allowance for the above, expression (4) takes the form

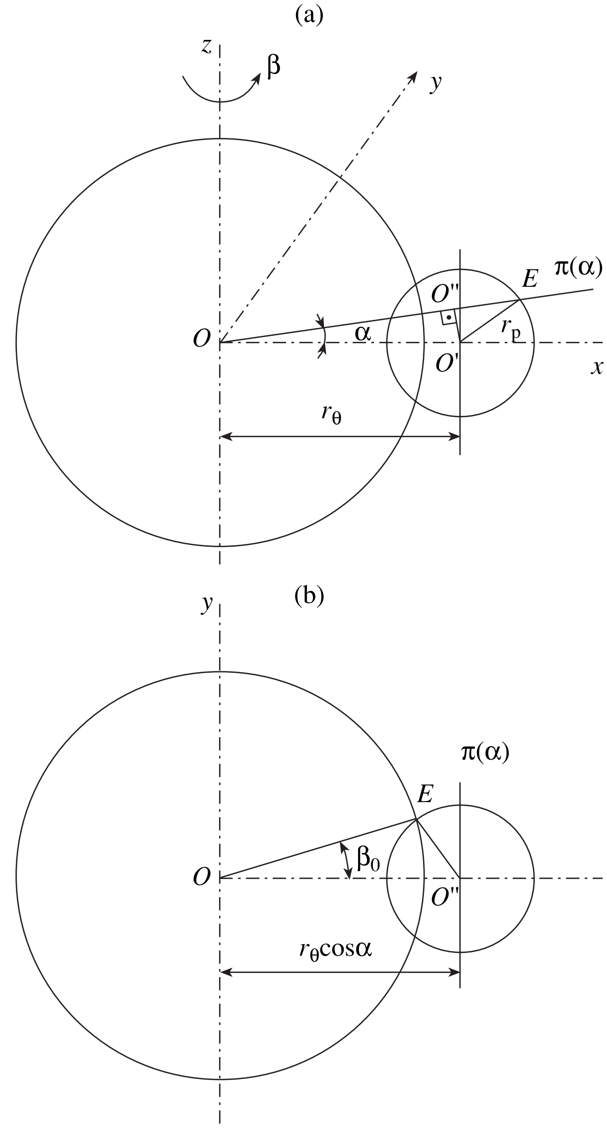

Fig. 2. Determination of the angle β_0 formed by the intersection of the bubble and the particle (the large and small spheres, respectively): the views in the planes (a) zOx and (b) $\pi(\alpha)$.

$$E = \frac{4}{R_b^2}\int_0^{b_c} s\,ds \int_{\theta_0}^{\pi/2} p(s)\,d\theta. \quad (15)$$

The equations of streamlines past the bubble for the potential and Stokes bubble rise patterns have the form [6, 7]

$$C_{pot} = \frac{1}{2}\frac{r_\theta^2}{R_b^2}\left(1 - \frac{R_b^3}{r_\theta^3}\right)\sin^2\theta, \quad (16)$$

$$C_{St} = \frac{1}{2}\frac{r_\theta^2}{R_b^2}\left(1 - \frac{3}{2}\frac{R_b}{r_\theta} + \frac{1}{2}\frac{R_b^3}{r_\theta^3}\right)\sin^2\theta. \quad (17)$$

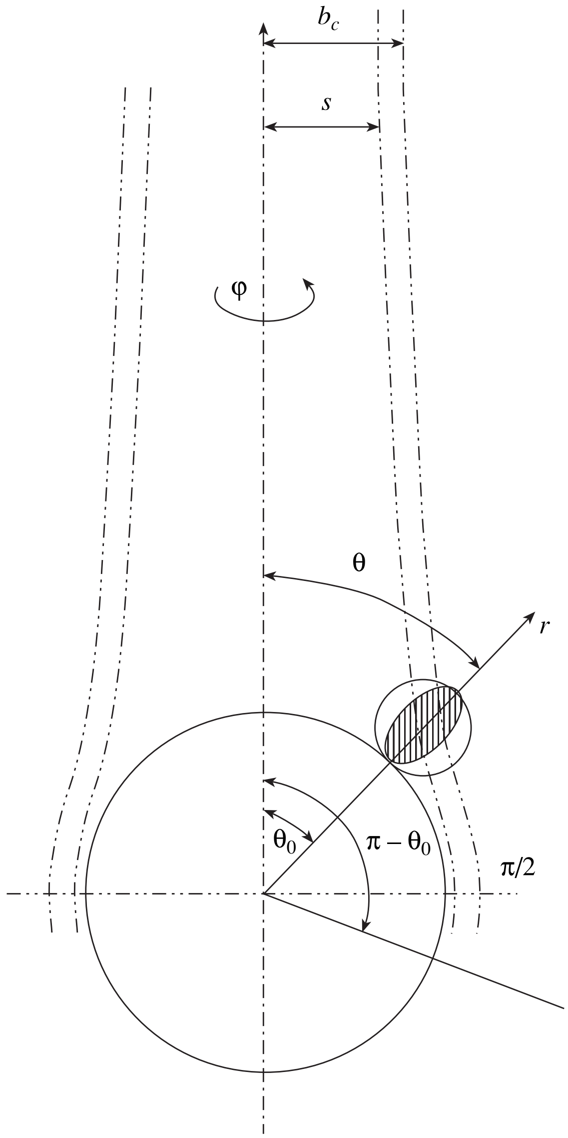

Fig. 3. Motion of the particle along streamlines past the bubble; (r, θ, φ) is the orthogonal spherical coordinate system fixed to the center of the bubble.

Hence, the angles θ_0 for the corresponding bubble rise patterns are found:

$$\theta_{0\text{pot}} = \arcsin\frac{R_\text{b}\sqrt{2C_\text{pot}}}{(R_\text{b}+r_\text{p})\sqrt{1-\dfrac{R_\text{b}^3}{(R_\text{b}+r_\text{p})^3}}}, \quad (18)$$

$$\theta_{0\text{St}} = \arcsin\frac{R_\text{b}\sqrt{2C_\text{St}}}{(R_\text{b}+r_\text{p})\sqrt{1-\dfrac{3}{2}\dfrac{R_\text{b}}{(R_\text{b}+r_\text{p})}+\dfrac{R_\text{b}^3}{2(R_\text{b}+r_\text{p})^3}}}. \quad (19)$$

The r_θ values can also be found from the equations of streamlines as the roots of the corresponding cubic equations:

$$r_{\theta\text{pot}}^3 - \frac{2R_\text{b}^3C_\text{pot}}{\sin^2\theta}r_{\theta\text{pot}} - R_\text{b}^3 = 0, \quad (20)$$

$$r_{\theta\text{St}}^3 - \frac{3R_\text{b}}{2}r_{\theta\text{St}}^2 - \frac{2R_\text{b}^2C_\text{St}}{\sin^2\theta}r_{\theta\text{St}} + \frac{R_\text{b}^3}{2} = 0. \quad (21)$$

Further, we calculate the entrainment efficiency E by formula (15) for the potential and Stokes bubble rise patterns.

Potential bubble rise pattern. Substitution of $r_{\theta\text{pot}}^3$ from expression (20) into expression (14) with the subsequent transformations yields

$$p(s) = -\frac{10C_\text{pot}}{3\pi\sin^2\theta}\frac{R_\text{b}}{r_\text{p}} + 2\frac{r_\theta}{\sqrt{R_\text{b}r_\theta}}\left(1-\frac{r_\theta}{3R_\text{b}}\right). \quad (22)$$

$r_{\theta\text{pot}}$ is the root of cubic equation (20) and is given by the formula [8]

$$r_{\theta\text{pot}} = \frac{R_\text{b}}{\sqrt[3]{2}}\left(\sqrt[3]{1+\sqrt{1-\frac{32C_\text{pot}^3}{27\sin^6\theta}}}+\sqrt[3]{1-\sqrt{1-\frac{32C_\text{pot}^3}{27\sin^6\theta}}}\right). \quad (23)$$

Let us estimate the quantity $C_\text{pot}/\sin^2\theta$ from streamline equation (16) and also evaluate the term $32C_\text{pot}^3/27\sin^6\theta$ in expression (23) at $r_\theta \in [R_\text{b}, R_\text{b} + r_\text{p}]$ ($r_\text{p} \le 0.1R_\text{b}$):

r_θ/R_b	1.00	1.01	1.03	1.05	1.07	1.09	1.10
$C_\text{pot}/\sin^2\theta$	0	0.015	0.045	0.075	0.105	0.135	0.150
$(32C_\text{pot}^3/27\sin^6\theta)\times10^3$	0	0.004	0.110	0.500	1.40	2.90	4.00

Hence, it follows that the root $\sqrt{1-\dfrac{32C_\text{pot}^3}{27\sin^6\theta}}$ in expression (23) is equal to 0.998–1.00 ≈ 1 within the r_θ range considered; in this case,

$$r_{\theta\text{pot}} \cong R_\text{b}. \quad (24)$$

In the neighborhood of the equatorial section of the bubble, where the small sphere touches the bubble surface (at $\theta \cong \pi/2$), the value of the constant C_pot in expression (22) for the probability $p(s)$ is

$$C_\text{pot} = \frac{1}{2}\frac{r_\theta^2}{R_\text{b}^2}\left(1-\frac{R_\text{b}^3}{r_\theta^3}\right).$$

Considering $r_\theta = R_b + r_p$, expanding the last expression into a Taylor series in powers of the small parameter (r_p/R_b), and truncating the series to the third-order terms, we obtain

$$C_{pot} = \frac{3}{2}\frac{r_p}{R_b}. \quad (25)$$

Since the function C_{pot} in the considered range varies linearly from 0 to $\frac{3}{2}\frac{r_p}{R_b}$, its average value over this range is $\frac{3}{4}\frac{r_p}{R_b}$. This C_{pot} value is used in calculating the probability $p(s)$ by expression (22).

The quantity θ_0, which is the lower limit in the inner integral in expression (15) and is determined for the potential bubble rise pattern by expression (18), can also be represented by expanding this function into a series in powers of the small parameter:

$$\theta_{0pot} = \arcsin\left[1 - \left(\frac{r_p}{R_b}\right)^2\right] = \frac{\pi}{2} - \sqrt{2}\frac{r_p}{R_b}. \quad (26)$$

In this expansion, the constant C_{pot} involved in expression (18) is taken to be equal to its maximal value $\frac{3}{2}\frac{r_p}{R_b}$ since its value is fixed at the point at which the small and large spheres (the particle and the bubble) touch each other, i.e., at $\theta = \theta_0$ and $r_\theta = R_b + r_p$.

Substitution of the obtained results into the inner integral in expression (15) gives

$$\int_{\theta_{0pot}}^{\pi/2}\left(-\frac{5}{2\pi\sin^2\theta} + \frac{4}{3}\right)d\theta = \frac{4\sqrt{2}}{3}\frac{r_p}{R_b}$$
$$- \tan\left(\sqrt{2}\frac{r_p}{R_b}\right)\frac{5}{2\pi} \approx \sqrt{2}\frac{r_p}{R_b}\left(\frac{4}{3} - \frac{5}{2\pi}\right).$$

Let us make the change of variable $s = r_\theta\sqrt{1 - \frac{R_b^3}{r_\theta^3}}$ in the outer integral of expression (15), i.e., express the current distance of the particle from the axis of motion of the bubble from the streamline equation; in this case,

$$sds = \frac{1}{2}d(s^2) = \left(r_\theta + \frac{R_b^3}{2r_\theta^2}\right)dr_\theta.$$

The limits of integration in the outer integral are also changed: $s = 0$ is replaced by $r_\theta = R_b$, and $s = b_c$, by $r_\theta = R_b + r_p$.

As a result, expression (15) for the efficiency of entrainment of the disklike particle by the rising bubble for the potential bubble rise pattern takes the form

$$E_{pot} = \frac{4}{R_b^2}\int_{R_b}^{R_b + r_p}\sqrt{2}\frac{r_p}{R_b}\left(\frac{4}{3} - \frac{5}{2\pi}\right)\left(r_\theta + \frac{R_b^3}{2r_\theta^2}\right)dr_\theta$$
$$= \left(8\sqrt{2} - \frac{15}{\pi}\right)\frac{r_p^2}{R_b^2} \approx 6.54\frac{r_p^2}{R_b^2}. \quad (27)$$

Stokes bubble rise pattern. To express $r_{\theta St}$ from Eq. (21), we transform this equation to a reduced cubic equation by making the change of variable $y = r_{\theta St} - R_b/2$. Since the resulting equation

$$y^3 - \left(\frac{2C_sR_b^2}{\sin^2\theta} + \frac{3R_b^2}{4}\right)y + \frac{R_b^3}{4} - \frac{C_{St}R_b^3}{\sin^2\theta} = 0 \quad (28)$$

has a negative discriminant, its solutions are three real roots [8]

$$y_1 = -\frac{2R_b}{\sqrt{3}}\sqrt[3]{\left(\frac{3}{4} + \frac{2C_{St}}{\sin^2\theta}\right)}\cos\frac{\varphi}{3},$$

$$y_2 = -\frac{2R_b}{\sqrt{3}}\sqrt[3]{\left(\frac{3}{4} + \frac{2C_{St}}{\sin^2\theta}\right)}\cos\left[\left(\frac{\varphi}{3}\right) + \frac{2\pi}{3}\right],$$

$$y_3 = -\frac{2R_b}{\sqrt{3}}\sqrt[3]{\left(\frac{3}{4} + \frac{2C_{St}}{\sin^2\theta}\right)}\cos\left[\left(\frac{\varphi}{3}\right) + \frac{4\pi}{3}\right].$$

Here, $\cos\varphi = \dfrac{\frac{1}{4} - \frac{C_{St}}{\sin^2\theta}}{2\sqrt{\left(\frac{3}{4} + \frac{2C_{St}}{\sin^2\theta}\right)^3\frac{1}{27}}}$ [8].

Estimating $2C_{St}/\sin^2\theta$ from streamline equation (17) as above for the potential bubble rise pattern, we find that this quantity varies from 0 (at $r_\theta/R_b = 1.0$) to 0.0145 (at $r_\theta/R_b = 1.1$). In this case, $\cos(\varphi/3)$ varies from 1 to 0.994, respectively.

Using this estimate, we calculate the roots y_i ($i = 1, 2, 3$) of Eq. (28) and, making the inverse change of variable $r_\theta = y_i + R_b/2$, obtain

$$r_{\theta St1} = -0.5R_b,$$

$$r_{\theta St2} = (1.0\ldots1.1)R_b,$$

$$r_{\theta St3} = (0.9\ldots1.0)R_b.$$

As a solution that is consistent with the physical nature of the problem under consideration ($R_b \le r_\theta \le R_b + r_p$), one can take the second root or the maximal value of the third root; i.e.,

$$r_{\theta St} \cong R_b. \quad (29)$$

The C_{St} value is estimated from streamline equation (17) just as the constant C_{pot} for the potential bubble rise pattern. At $\theta = \theta_{0St}$ and $r_\theta = R_b + r_p$, the C_{St} value is determined to within the third order of smallness by the expression

$$C_{St} = \frac{3}{2}\frac{r_p^2}{R_b^2}. \quad (30)$$

The angle θ_{0St}, which is determined for the Stokes bubble rise pattern by expression (19), can be expanded into a series in powers of the argument of the arcsine function

$$\theta_{0St} = \arcsin\left(1 - \frac{r_p}{R_b}\right). \quad (31)$$

The subsequent expansion of the function itself is performed not at the zero point ($r_p/R_b \longrightarrow 0$) (since, at this point, the function $\arcsin(1 - r_p/R_b)$ has the infinite ($-\infty$) first derivative) but in its neighborhood ($r_p/R_b = 0.1$) [9]:

$$f\left(\frac{r_p}{R_b}\right) = \arcsin\left(1 - \frac{r_p}{R_b}\right) = f(0.1) + \frac{f'(0.1)}{1!}\left(\frac{r_p}{R_b} - 0.1\right) + \frac{f''(0.1)}{2!}\left(\frac{r_p}{R_b} - 0.1\right)^2 = 1.40 - 3.38\frac{r_p}{R_b} + 5.43\frac{r_p^2}{R_b^2}. \quad (32)$$

Substitution of the obtained results into the inner integral in expression (15) gives

$$\int_{\theta_{0St}}^{\pi/2}\left[\frac{5(R_b^3 - r_\theta^3)}{3\pi r_p R_b r_\theta} + 2\frac{r_\theta^2}{\sqrt{R_b r_\theta}}\left(\frac{1}{r_\theta} - \frac{1}{3R_b}\right)\right]d\theta = 0.667\theta\Big|_{\theta_{0St}}^{\pi/2} = 0.113 + 2.253\frac{r_p}{R_b} - 3.620\frac{r_p^2}{R_b^2}. \quad (33)$$

In the outer integral in expression (15), we also make the change of variable s by the function r_θ from the streamline equation:

$$s = r_\theta\sqrt{1 - \frac{3}{2}\frac{R_b}{r_\theta} + \frac{1}{2}\frac{R_b^3}{r_\theta^3}}.$$

Here, $sds = \left(r_\theta - \frac{3R_b}{4} - \frac{R_b^3}{4r_\theta^2}\right)dr_\theta$.

The limits of integration in the outer integral are changed much as for the potential bubble rise pattern.

The total efficiency of flotation entrainment of the disklike particle by the rising bubble for the Stokes bubble rise pattern is

$$E_{St} = \frac{4}{R_b^2}\int_{R_b}^{R_b + r_p}\left(0.113 + 2.253\frac{r_p}{R_b} - 3.62\frac{r_p^2}{R_b^2}\right) \times \left(r_\theta - \frac{3}{4}R_b - \frac{R_b^3}{4r_\theta^2}\right)dr_\theta \approx 0.23\frac{r_p^2}{R_b^2}. \quad (34)$$

Since, in the Stokes bubble rise pattern, along with the random (equiprobable) orientation of the disklike particle with respect to streamlines past the bubble, a certain fixed orientation is also possible, we estimate the maximal entrainment efficiency, at which the plane of the disklike particle is perpendicular to streamlines.

Using the mathematical model accepted in this work, the probability $p_\perp(s)$ of the disklike particle moving with respect to the bubble perpendicularly to streamlines will be considered as the double ratio of the area F_s of the curved segment (Fig. 4, hatched region) cut from the disk with the bubble surface to the area F_d of the disk:

$$p_\perp(s) = \frac{2F_s}{F_d}.$$

To estimate $p_\perp(s)$ with allowance for the actual ratio between the sizes of the bubble and the particle ($R_b \gg r_p$), the curved segment area F_s can be replaced with sufficient accuracy by the area of the ordinary segment that is bounded by the arc of the radius r_p and has the central angle 2γ (Fig. 4). The area F_s' of the latter segment in the accepted notation is [8]

$$F_s' = \frac{r_p^2}{2}(2\gamma - \sin 2\gamma).$$

In this case, $p_\perp(s) = \frac{2\gamma - \sin 2\gamma}{\pi}$.

The angle γ is found from ODO' by the law of cosines (Fig. 4)

$$\gamma = \arccos\frac{r_\theta^2 + r_p^2 - R_b^2}{2r_\theta r_p}.$$

Substitution of the γ value into the expression for $p_\perp(s)$ with subsequent transformations similar to those performed in determining $p(s)$ yields

$$p_\perp(s) = 1 - \frac{r_\theta^2 + r_p^2 - R_b^2}{2\pi r_\theta r_p}. \quad (35)$$

Further, we calculate the entrainment efficiency $E_{\perp St}$ from expression (15) using the previously obtained

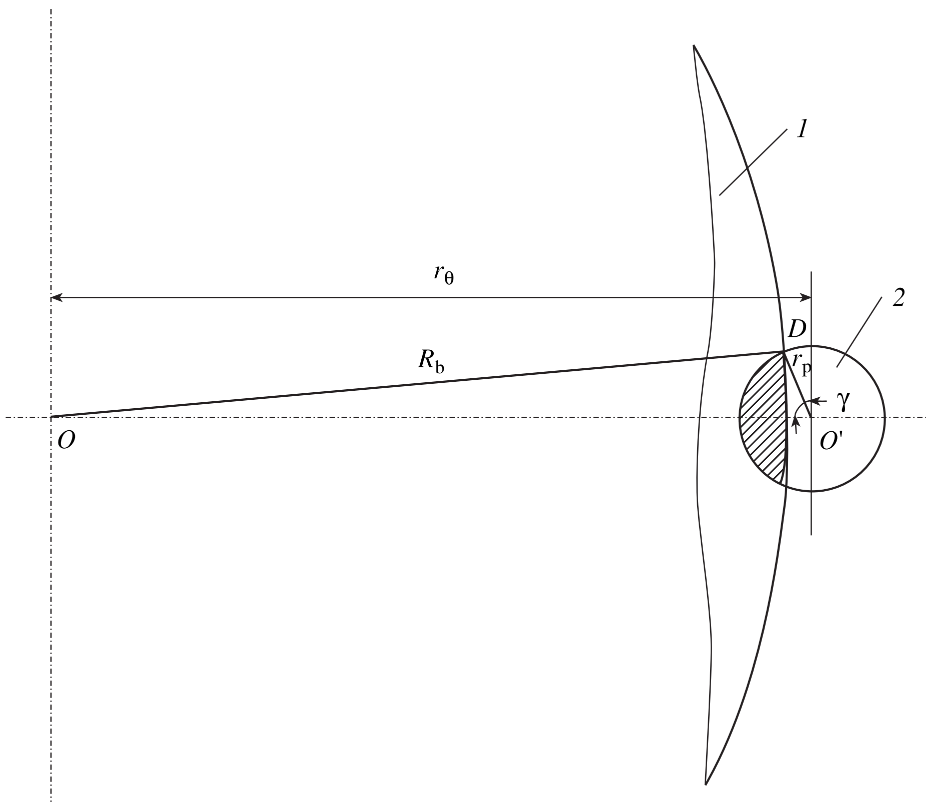

Fig. 4. Position of bubble *1* and disklike particle *2* oriented perpendicularly to streamlines (the streamlines are normal to the plane of the figure).

expressions for r_θ and θ_{0St} (expressions (29) and (32), respectively). Similarly to expression (34), the total efficiency of flotation entrainment of the disklike particle oriented perpendicularly to streamlines by the rising bubble for the Stokes bubble rise pattern is

$$E_{\perp St} = \frac{4}{R_b^2}\int_{R_b}^{R_b+r_p}\left(0.17 + 3.37\frac{r_p}{R_b}\right) \times \left(r_\theta - \frac{3}{4}R_b - \frac{R_b^3}{4r_\theta^2}\right)dr_\theta \approx 0.34\frac{r_p^2}{R_b^2}. \quad (36)$$

Figure 5 presents the curves of the efficiency of inertialess flotation entrainment of disklike and spherical particles for the potential and Stokes bubble rise patterns. The calculations were performed at the same characteristic sizes of particles and bubbles by above expressions (27), (34), and (36) for disklike particles and by known expressions (1) and (2) for spherical particles for the corresponding flow patterns.

As follows from Fig. 5, at the same characteristic sizes (r_p/R_b), the efficiency of entrainment of disklike particles is much lower than that for spherical ones. This result is consistent with the physical nature of the considered process: the probability that a spherical particle touches the bubble for a particle whose distance from the axis of motion of the bubble is smaller than the critical distance b_c is unity, whereas this probability for a disklike particle is always lower than unity and is determined by the orientation of the particle with respect to the bubble surface. For example, if the plane of disklike particles is aligned with streamlines past the bubble, the probability that they touch the bubble surface (that they are entrained by the bubble) is close to zero. The efficiency of entrainment of a disklike particle oriented perpendicularly to streamlines past the bubble in the case of the Stokes bubble rise pattern is half as high again as the entrainment efficiency in the case of the random orientation of the same particle ($E_{\perp St}/E_{St}$ = 0.34/0.23 ≈ 1.48). This is also logically consistent with the physical nature of the process considered.

The mathematical model used in this work for the probabilistic definition of the orientation of dispersed-phase particles with respect to the surface of an entraining particle (bubble, drop, filtration bed grain, etc.) can also be applied to calculate the efficiency of entrainment (settling) of other classes of anisometric particles (flat two-dimensional particles with two characteristic

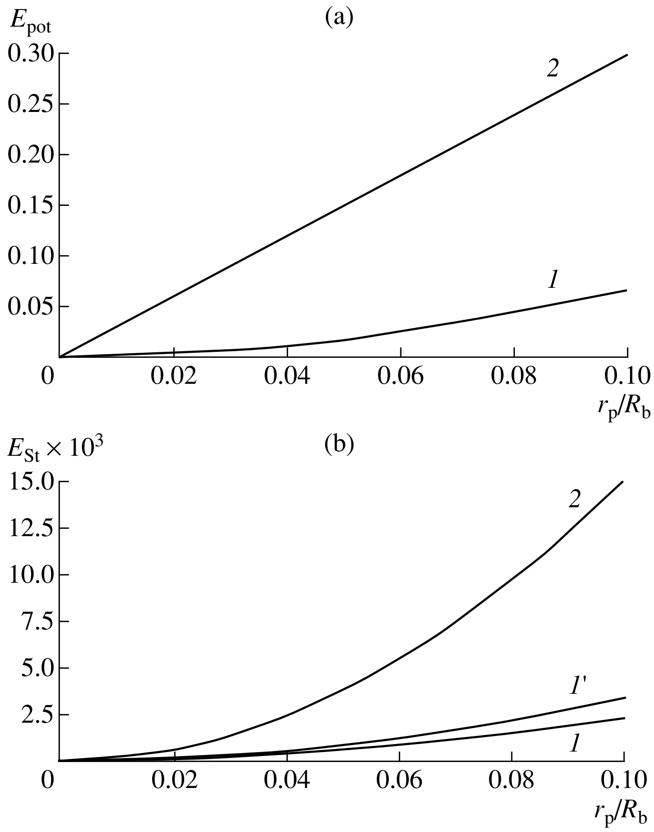

Fig. 5. Efficiency of inertialess flotation of (*1*) randomly (equiprobably) oriented disklike particles, (*1*') disklike particles oriented perpendicularly to streamlines in the neighborhood of the bubble, and (*2*) spherical particles for the (a) potential and (b) Stokes bubble rise patterns.

sizes, e.g., elliptic particles; three-dimensional particles with three characteristic sizes).

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (project no. 03-01-00620).

NOTATION

b_c—limiting distance from the axis of motion of the bubble to such a streamline that a particle on this streamline can touch the bubble surface, m;

C—constant for streamlines equidistant from the axis of motion of the rising bubble;

E—flotation entrainment efficiency;

F—disk area, m^2;

$p(s)$—probability that a particle touches the bubble;

R_b, r_p—bubble and particle radii, respectively, m;

r_θ—distance between the centers of the bubble and the particle, m;

s—current distance from the axis of motion of the bubble to a streamline ($0 \le s \le b_c$), m;

θ_0—angle between the equatorial plane of the bubble and the point at which a particle moving along a streamline touches the bubble surface, rad.

SUBSCRIPTS AND SUPERSCRIPTS

pot—potential bubble rise pattern;

St—Stokes bubble rise pattern.

REFERENCES

1. Dukhin, S.S., Rulev, N.N., and Dimitrov, D.S., *Koagulyatsiya i dinamika tonkikh plenok* (Coagulation and Dynamics of Thin Films), Kiev: Naukova Dumka, 1986.
2. Deryagin, B.V., Dukhin, S.S., and Rulev, N.N., *Mikroflotatsiya: vodoochistka, obogashchenie* (Microflotation: Water Treatment and Beneficiation), Moscow: Khimiya, 1986.
3. Sutherland, K.L., Physical Chemistry of Flotation: XI. Kinetics of the Flotation Process, *J. Phys. Colloid Chem.*, 1948, vol. 52, no. 2, pp. 394–425.
4. Sokovnin, O.M. and Polovnikov, V.A., Efficiency of Flotation Entrainment of Rodlike Particles by a Bubble Rising in a Potential Flow, *Kolloidn. Zh.,* 1993, vol. 55, no. 3, pp. 161–166.
5. Sokovnin, O.M. and Polovnikov, V.A., Calculation of the Efficiency of Entraining Rodlike Particles by a Bubble Rising in a Stokes Flow, *Teor. Osn. Khim. Tekhnol.*, 1996, vol. 30, no. 2, pp. 213–216 [*Theor. Found. Chem. Eng.* (Engl. Transl.), vol. 30, no. 2, pp. 192–195].
6. Loitsyanskii, L.G., *Mekhanika zhidkosti i gaza* (Fluid Mechanics), Moscow: Nauka, 1978.
7. Kutepov, A.M., Polyanin, A.D., Zapryanov, Z.D., and Vyaz'min, A.V., *Khimicheskaya gidrodinamika. Spravochnoe posobie* (Chemical Hydrodynamics: Handbook), Moscow: Byuro Kvantum, 1996.
8. Bronshtein, I.N. and Semendyaev, K.A., *Spravochnik po matematike dlya inzhenerov i uchashchikhsya vuzov* (Handbook of Mathematics for Engineers and Students), Moscow: Nauka, 1980.
9. Fikhtengol'ts, G.M., *Osnovy matematicheskogo analiza* (Principles of Calculus), Moscow: Nauka, 1968, vol. 1.