Instability of the wet cube cone soap film

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Abstract

A “dry” conical soap film on a cubical frame is not stable. Recent experimental evidence seems to indicate that adding liquid to form “Plateau borders” stabilizes the conical film, perhaps to arbitrarily low liquid volumes. This paper presents numerical simulation evidence that the the wet cone is unstable for low enough liquid volume, with the critical liquid volume fraction being about 0.000274.

1 Introduction

A “dry” soap film is the idealization that a soap film has no thickness. Jean Taylor [T] showed that the only singularities possible in dry 2D films in 3D space are three films meeting along a curve at 120° and four triple curves meeting at a point, as in the cone over a tetrahedral frame. Hence the conical film over a cubical frame (figure 1A) is unstable and jumps to a film with tetrahedral singular points (figure 1B).

Real films are “wet” in that they have some thickness and have Plateau borders. A Plateau border is a thickening of the film where several sheets meet, forming concave triangular tubes in place of the triple junctions of dry films. The surface tension on a Plateau border is half that of a dry film, since a dry film is really a double layer. A physical film on a cubical frame that starts wet (figure 1C) rapidly drains by gravity to a nearly dry film and jumps to figure 1B. But recent laboratory experiments by Weaire and Phelan [WP] seem to show that keeping the Plateau borders from drying out by constantly feeding liquid in from the top stabilizes the wet cone. It is consistent with all these facts that a wet cone is stable for any liquid volume, but that the barrier to jumping to the figure 1B form is low enough that real films rapidly cross it when draining, due to asymmetries or fluctuations.

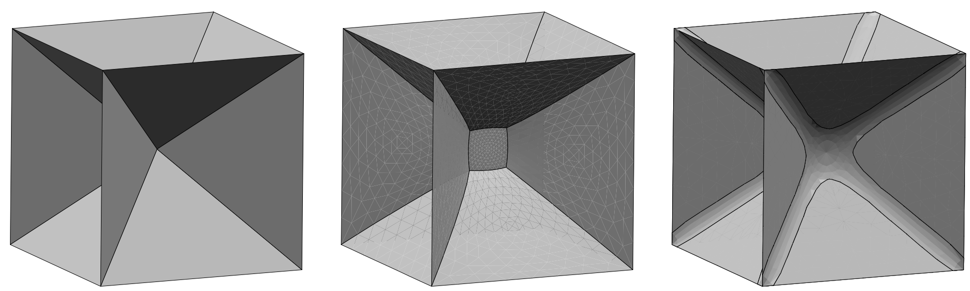

Figure 1: A. Dry cube cone film. B. Stable dry film. C. Wet cone film.

The question considered in this paper is whether the wet cube cone film (figure 1C) is stable. The model used is still an idealization to the extent that all the liquid is in the Plateau borders, the "dry" part of the film has zero thickness, the wires of the cubical frame have zero diameter, and there are no dynamical effects due to flowing liquid. Also, gravity will be neglected.

There are two meaningful alternative conditions to impose on the Plateau borders: constant volume or constant pressure. Constant volume is natural when considering a film on an isolated cubical frame. However, if the film is seen as part of a larger system, a foam for example, then the large volume of liquid outside the cube acts as a constant pressure reservoir. It turned out that for the liquid fractions involved in this paper, there was no significant difference in stability between constant volume and constant pressure.

Drier wet cone films will be more unstable than wetter cones. This is because the Plateau borders are of nearly uniform width for most of their length, so drier cones essentially contain a shrunken copy of the wetter cone. Hence any unstable perturbation of the wetter cone may be scaled down to an unstable perturbation of the drier cone. Hence there is a critical liquid fraction V_{crit} above which the wet cone is stable and below which it is unstable. This is not a rigorous argument, but it is compelling enough for the purposes of this paper.

Numerical models are an obvious route to try to decide on stability. Weaire and Phelan [WP] used a very simple model to conclude stability to arbitrarily low liquid volume. But their model is far too simple to trust in a delicate case like this.

This paper uses the Surface Evolver [B2] computer program to provide two lines of evidence that the wet cube cone is unstable at low liquid volumes. The first line consists of modelling the wet cube cone and finding a deformation that decreases energy. This would seem to be straightforward, but turned out to have considerable difficulties due to the low critical volume fraction. The second line shows that the central portion of the wet cone in isolation is definitely unstable, and argues that the whole wet cone is still unstable for some low enough liquid volume. The second line is provided as a backup to the first line, and because it historically preceded the first line by a year and provided the motivation to persevere to find the second line.

Section 2 discusses related results in other dimensions. Section 3 does the deformations of the wet cube cone. Section 4 has the central section argument. Section 5 sketches out the stability diagram for the wet cube cone. Section 6 discusses wetting other cones.

2 Known results in other dimensions

For a 1D film spanning the corners of a square (figure 2), it is known [BW],[BM] that if the liquid area is greater than about 0.1 of the area of the square, then the wet cone (figure 2C) is stable. For smaller liquid area, the wet cone is unstable toward jumping to an asymmetric wet cone (figure 2D) or a film with two triangular Plateau borders (figure 2E).

In space dimension 4 and higher, it is known [B1] that the dry conical film over a hypercube frame is stable, in fact absolutely area minimizing. The moral of [B1] is that higher dimension stabilizes cones. Thus the wet cube cone should be more stable than the 2D wet cone, possibly even stable down to arbitrarily low liquid volume.

3 Surface Evolver model of the wet cube cone

The Surface Evolver [B2] is a computer program for modelling liquid surfaces shaped by various forces and constraints. A surface has a finite element representation as a set of triangles, with any number of triangles meeting along an edge so there are no restrictions on surface topology. Either linear or quadratic triangle elements may be used. A surface begins as a crude finite element representation, and is then evolved by refining the triangulation and minimizing energy by gradient descent (augmented with conjugate gradient). The Evolver can also calculate the Hessian matrix of the energy, viewing the energy as a function of all the vertex coordinates. The Hessian can be used to converge quickly to an equilibrium by Newton's Method, and can be used to test the stability of an equilibrium by testing the Hessian's positive definiteness. The Evolver can also calculate the lowest few eigenvalues of the Hessian, which is useful for seeking the critical point of transition between stability and instability. In order to be able to directly compare eigenvalues for different triangulations of a surface, the eigenvalues are defined with respect to an inner product which approximates the L_2 inner product of functions on the surface.

This section describes perturbations of the wet cube cone that decrease its

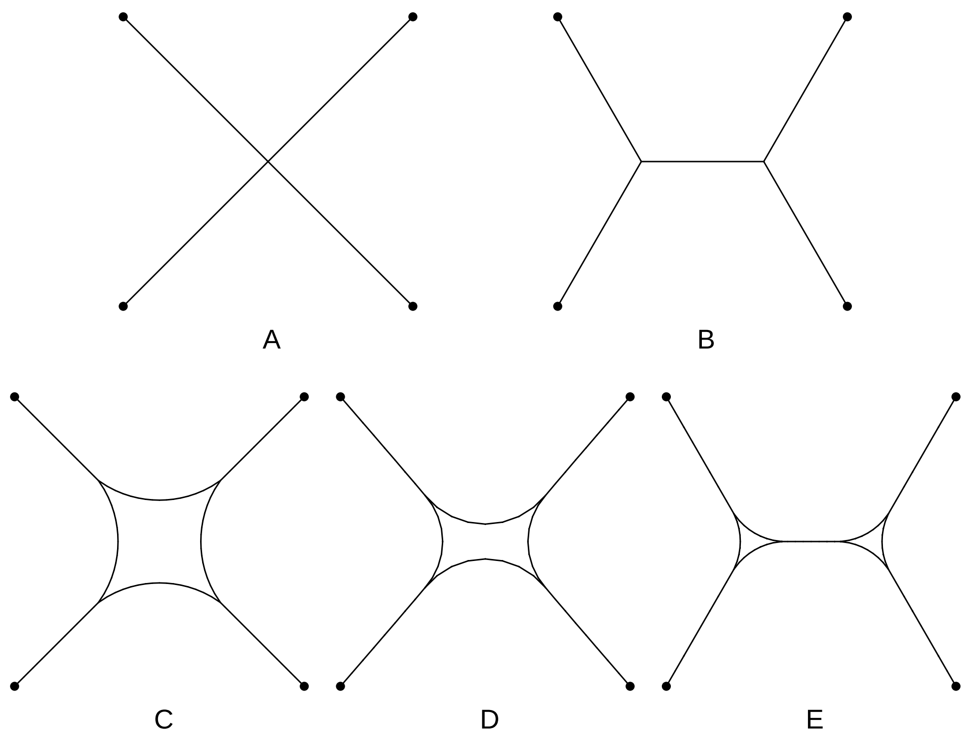

Figure 2: A. Unstable dry cone in the plane. B. Stable dry film. C. Wet cone at the critical volume for stability. D. Stable mode for liquid fraction slightly below critical. E. Stable mode for small liquid fractions. B, D and E may also be rotated 90 degrees.

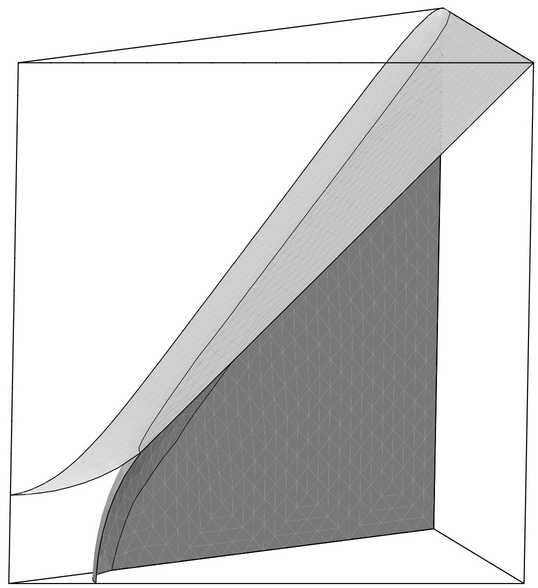

Figure 3: Fundamental region of the wet cube cone for 16-fold symmetry. The front plane of the bounding prism is the $y = 0$ symmetry plane, the back is $x = y$, the bottom is $z = 0$, the top is $z = 1$, and the right is $x = 1$. The volume fraction shown is 0.02.

energy. The wet cube cone shares the 48-fold symmetry of the cube, but the presumed perturbation towards a wet version of the dry cube film (figure 1B) still has 16-fold symmetry. Thus it suffices to make a Surface Evolver model of a fundamental region for the perturbation, shown in figure 3. The perturbation involves warping the Plateau border and the dry film on the $x = z$ plane and flattening the volume near the origin while the dry film on the $x = y$ plane remains flat. Figure 8 shows a perturbation to the point that opposite films touch at the center.

To guarantee perfect symmetry in the starting configuration, the 16-fold fundamental region was constructed by first evolving a 48-fold fundamental region and then triplicating it. The contribution to the energy of flat film on the $x = y$ plane was replaced by a line integral along its boundary to reduce the computational burden.

The obvious way to test the stability of the 16-fold fundamental region is to let the Evolver calculate the lowest eigenvalue of its energy Hessian. However, calculating Hessian eigenvalues is not perfectly straightforward, as tangential motions of the vertices give many near-zero eigenvalues, even negative eigenval-

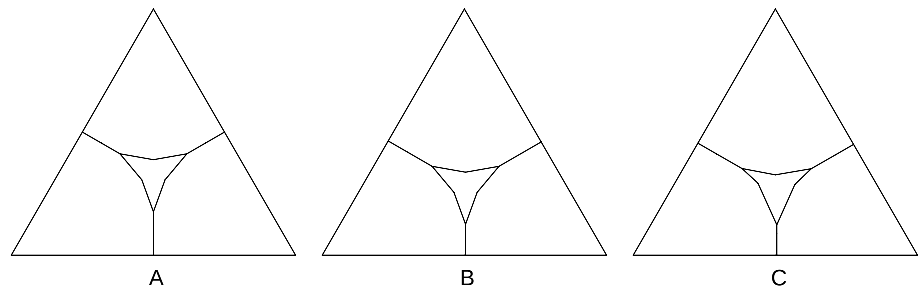

Figure 4: Distortion of Plateau border by normal motion. A: Undistorted version. B: Undistorted after motion. C: Distorted after normal motion.

ues in a seemingly well-evolved surface. It is extraordinarily difficult to evolve a highly refined surface to the point of having a positive definite Hessian. Therefore the Evolver has a mode in which only perturbations of the vertices normal to the surface are permitted; these are the perturbations that count in a smooth surface in any case. Unfortunately, the normal-mode Hessian did not show instability, even for volume fractions down to 0.000008 and highly refined surfaces. The lowest eigenvalues looked to be heading toward zero with increasing refinement, but never got there.

The difficulty turned out to be that the energy-decreasing perturbation really had enough tangential component that forcing it to follow normal vectors caused enough distortion to eliminate the instability. The problem is illustrated in figure 4, which shows a very coarsely refined wet triple junction with free boundaries on a fixed triangular frame. Without normal mode, the film in figure 4A is in neutral equilibrium, since it may be translated within the triangle with exactly equal energy (figure 4B). But with normal mode, trying to translate using only normal motions distorts the Plateau border, raising the energy (figure 4C). The lowest eigenvalue here happens to be 0.0723. A little dimensional analysis shows that eigenvalue grows inversely with the diameter of the Plateau border, so the problem gets worse and worse with smaller liquid volume.

The cure turned out to be to define a direction of motion in the direction the perturbation wanted to go, rather than the normal direction. I added a Hessian mode to the Evolver that allows motion only in the $(1, 1, -2)$ direction at each vertex. This suppressed all the unwanted tangential motion, but allowed the desired perturbation. This mode finally revealed negative eigenvalues in a film with liquid fraction 0.00018, showing the wet cube cone can be unstable.

To determine the critical liquid fraction somewhat accurately, the liquid fraction 0.00018 film had its wire frame effectively shrunk by immobilizing vertices with $x + z > c$. The constant was lowered from $c = 2$ until stability returned.

$x + z$ cutoff	4392 facets	17568 facets
2.00	-0.2397	-0.2523
1.70	-0.0218	-0.0246
1.69	0.0015	-0.0129
1.68	0.0015	-0.0098
1.67	0.0072	0.0080

Table 1: Eigenvalues for successive refinements with vertices immobilized for $x + z \geq c$, for volume fraction 0.00018.

volume fraction	facets	eigenvalue
0.000272	1254	0.04160
	4916	0.01371
	19664	-0.00072
0.000274	1254	0.04337
	4916	0.02392
	19664	0.01061

Table 2: Eigenvalues for successive refinements of full wet cube regions.

Table 1 shows the results for two successive refinements of the film. Both films used quadratic elements for better accuracy. Taking the critical c to be 1.67, the critical liquid fraction is about

$$V_{crit} \approx 0.00018 \left(\frac{2.00}{1.67} \right)^2 \approx 0.00026. \qquad (1)$$

Immobilizing vertices for $x + z \geq c$ is not quite the same as having the cube wire there, since facets may extend beyond $x + z = c$, but it should be pretty close for a highly refined surface.

Wet cube cones near the critical volume fraction were tested for stability. The most relevant results are given in table 2. The facets are all quadratic, and the direction of perturbation was changed from $(1, 1, -2)$ to $(x, y, -2z)$, which gave slightly lower eigenvalues. It appears that the best estimate of the critical volume fraction is 0.000274.

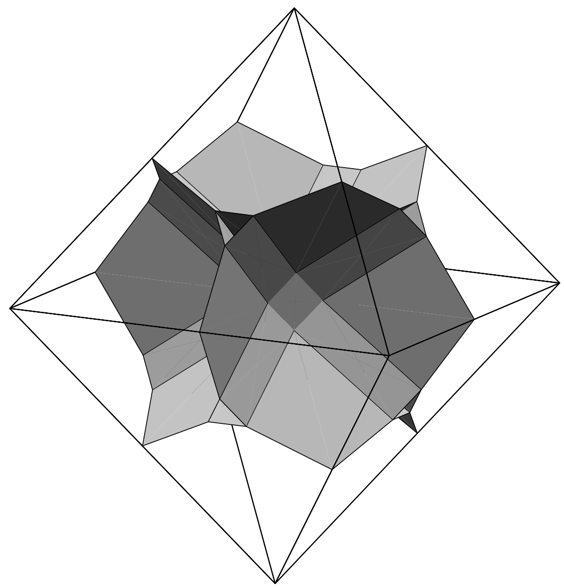

Figure 5: Central region with free boundaries on an octahedron, in symmetric equilibrium. Not evolved.

4 Instability of central region with free boundary

Since modelling a full wet cube cone (or even a fundamental region of it) is so difficult, I originally decided to try to isolate the inner part of the wet cone from the outer parts. The inner part should be pretty much the same for all liquid volumes if scaled correctly, and the outer parts should be just small perturbations of flat planes and Plateau borders with cylindrical surfaces. To that end, I truncated the inner portion with an octahedron (figure 5), with the film boundary free to move on the octahedron. I figured this would be the least disruptive way to isolate the central region, in that for the dry film the central point can translate arbitrarily within the octahedron without area change, since the films remain orthogonal to the octahedron. The constraint on the Plateau borders is constant pressure, since even if the whole wet cone is at constant volume, the outer Plateau borders (being much larger than the central region) serve as a constant pressure reservoir.

Surface Evolver calculations show that this central region by itself is unstable. In fact, the eigenspace for the one negative eigenvalue has dimension 2. This might seem strange, since there are three ways for the cone to change

Pressure	Facets	Coefficient	Critical volume
2	21	-0.8211	
	84	-0.8605	
	336	-0.8821	
	1219	-0.8905	0.000055
4	37	-0.9504	
	148	-1.0729	
	592	-1.1100	
	2323	-1.1383	0.000065
8	57	-1.1653	
	228	-1.3477	
	819	-1.4338	
	3268	-1.5205	0.000066

Table 3: Coefficients of quadratic term in energy as function of perturbation of fixed boundary. Critical volume values are from equation 13.

to figure 1B, but they turn out to be linearly dependent. The infinitesimal generators of the three ways are essentially like

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

To get more precise control over the boundary on the octahedron, in order to match it to perturbations of the outer regions, I made a version of the central region with a fixed boundary trace on the octahedron. This trace consists of straight segments and circular arcs of radius $1/p$ around the Plateau border, where p is the difference of external pressure over internal pressure. This trace has an adjustable parameter in it that moves it in the direction of the unstable perturbation. This enables me to make a known perturbation of the form $(h, h, -2h)$ uniformly over the entire surface, re-minimize the surface using the Hessian and Newton's Method, and thus numerically measure the coefficient of the quadratic term in the energy as a function of h. The value of h actually used was 0.00001, small enough not to distort the triangulation. The results are presented in table 3.

So the question now is, is the area loss of the unstable deformations of the inner region compensated for by the area increase of the deformation of the outer regions connecting the inner region to the cube frame? I present the following argument (not a proof yet) that it isn't, for low enough volume.

The deformation of the inner region bends the large flat outer films away from flatness. For small deformations, this outer perturbation is nearly a harmonic function, since a nearly flat minimal surface is nearly the graph of a harmonic function. The dominant term in this harmonic function ought to be $\log(r)$, since the deformation is largest toward the center of the cube. If the cube frame is at distance D from the center, then the area excess of the deformation turns out to be proportional to $1/\log(D)$. Hence for large enough D, the central area decrease will win.

More precisely, consider the following specific deformation to connect the perturbed boundary to the outer frame. Let the inner octahedron face be $x + y + z = a$ and let the cube corner be at (D, D, D). Then the octahedron face $x + y + z = 2D$ is still within the cube, at least the part containing the film. Consider deformations that on the octahedron face $x+y+z = t$ slide the section by

$$h(t)(1, 1, -2), \qquad h(2D) = 0, \tag{3}$$

with the accompanying stretching and shrinking of the dry film on the face edges, as in figure 4B. The perturbation is taken to be zero between $x + y + z = 2D$ and the cube frame. Note this perturbation preserves Plateau border volume exactly, and on each slice the net change in cross-section length is zero. So the only area increase comes from the bending of the surface. The bending of the Plateau border contributes less excess than would the bending of the dry film (easy calculation), so we just compute the area excess of the dry film.

To flatten out the domain, make the change of variables to an orthonormal system of coordinate (u, v, w) defined by

$$u = (x + y + z)/\sqrt{3}, \qquad v = (x - 2y + z)/\sqrt{6}, \qquad w = (z - x)/\sqrt{2}. \tag{4}$$

The dry film subject to our perturbation is then defined by

$$a/\sqrt{3} \le u \le 2D/\sqrt{3}, \qquad 0 \le v \le u/\sqrt{2}, \qquad w = 0. \tag{5}$$

The magnitude of the perturbation is

$$w(u, v) = \frac{3}{\sqrt{2}} h(u). \tag{6}$$

The area is (since film is double density here)

$$A = 2\int_{a/\sqrt{3}}^{2D/\sqrt{3}} \int_0^{u/\sqrt{2}} \sqrt{1 + w_u^2}\, dv\, du, \tag{7}$$

so the excess area is

$$\begin{aligned} Excess &= 2\int_{a/\sqrt{3}}^{2D/\sqrt{3}} \int_0^{u/\sqrt{2}} \sqrt{1 + w_u^2} - 1\, dv\, du \\ &\approx 2\int_{a/\sqrt{3}}^{2D/\sqrt{3}} \int_0^{u/\sqrt{2}} \frac{1}{2} w_u^2\, dv\, du \\ &= \int_{a/\sqrt{3}}^{2D/\sqrt{3}} \frac{u}{\sqrt{2}} \frac{9}{2} h'(u)^2\, du. \end{aligned} \tag{8}$$

The calculus of variations shows that the optimal form of $h(u)$ here is

$$h(u) = h_0 \frac{\log(2D/\sqrt{3}) - \log u}{\log(2D/\sqrt{3}) - \log(a/\sqrt{3})}, \tag{9}$$

where $h_0 = h(a/\sqrt{3})$ is the perturbation at the inner octahedron. Plugging this back in gives

$$\begin{aligned} Excess &= \frac{9}{2\sqrt{2}} \int_{a/\sqrt{3}}^{2D/\sqrt{3}} u h_0^2 \frac{1}{u^2} (\log(2D/\sqrt{3}) - \log(a/\sqrt{3}))^{-2} du \\ &= \frac{9}{2\sqrt{2}} \frac{1}{\log(2D/a)} h_0^2. \end{aligned} \tag{10}$$

So if D is large enough, the excess in outer area is less than the area loss in the inner region. In particular, if we let C be the coefficient from table 3, with $a = 2$, then the critical value of D occurs when

$$C + \frac{9}{2\sqrt{2}} \frac{1}{\log D} = 0, \tag{11}$$

or

$$D = \exp(-9/2\sqrt{2}C). \tag{12}$$

To get the corresponding liquid fraction, multiply the length of the Plateau border, $\sqrt{3}D$, by its cross section for a given radius r, $(\sqrt{3} - \pi/2)r^2$, and divide by the cube octant volume:

$$V_{crit} = \frac{\sqrt{3}D(\sqrt{3} - \pi/2)r^2}{D^3} = \sqrt{3}(\sqrt{3} - \pi/2)r^2 \exp(9/\sqrt{2}C). \tag{13}$$

These numbers are in the last column of table 3. Given the restricted nature of the perturbations considered, these values only give a lower bound on the true critical volume. But they are only a factor of four lower than the best estimate of $V_{crit} = 0.000274$, so this approach does seem to provide a good picture of what is happening and why the critical volume fraction is so low. Whenever logs and exponentials appear, numbers can get extreme very easily.

5 Wet cube cone stability diagram

A stability diagram is a handy way to describe the behavior of a system. For simplicity, we first consider the system consisting of 1/16 of a wet cube cone. The control parameter will be the liquid volume fraction, and the response will be the asymmetry of the film, measured as the difference in the volumes of the two exterior regions of the film. This stability diagram will be rather qualitative, so exact definitions are not needed. The stability diagram is sketched in figure 6.

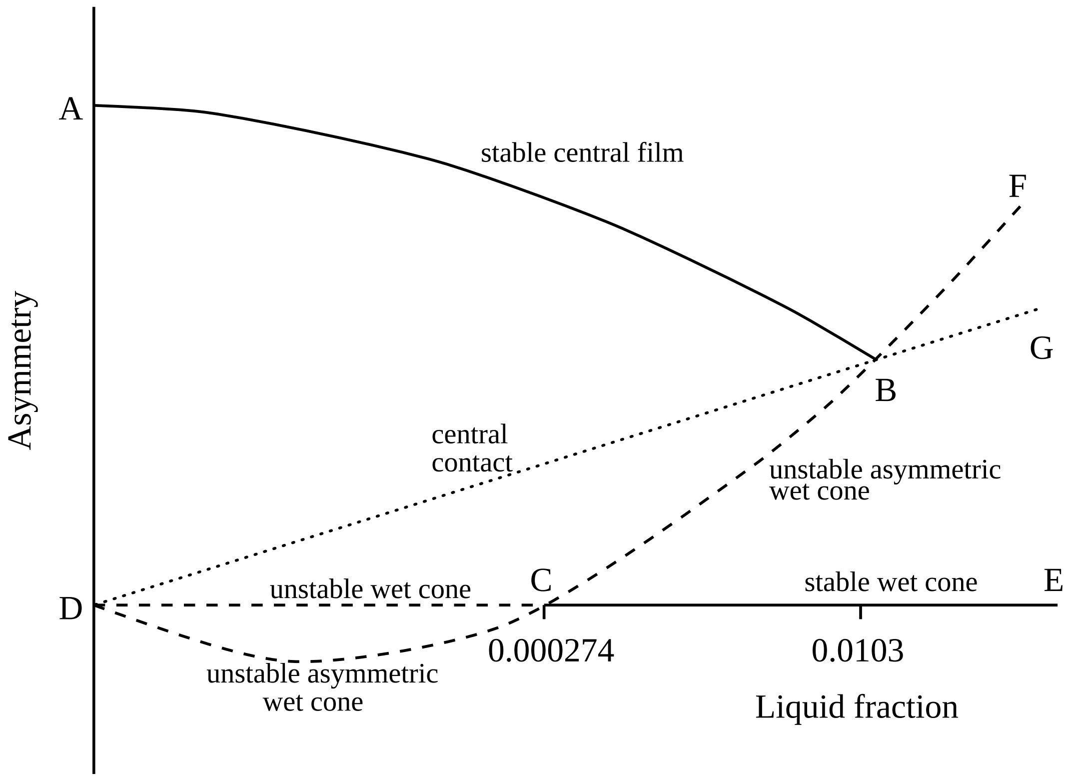

Figure 6: Qualitative stability diagram of wet cube film for fixed volume. Not to any kind of scale whatsoever.

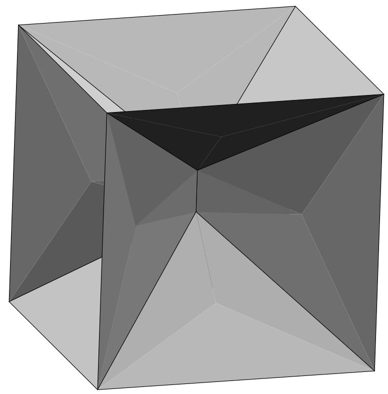

Figure 7: Dry film version of the wet films on the "other side" of the wet cone saddle point from the central square, i.e. the lower curve DC of figure 6. No dry film of this shape actually exists, however.

Solid lines represent stable films, and dashed lines represent unstable films. Point D is the dry cube cone, figure 1A. Point C is the critical volume symmetric wet cube cone. The dashed straight line DC represents the unstable symmetric wet cube cones. The solid line CE represents the stable wet cones. Point A is the dry cube film, figure 1B. The curve AB represents progressive wetting of the the dry cube film, until it reaches the critical point B, where the central film disappears. The dotted line DBG is the transition to a central film, where opposite sides of the central region first touch. These are not equilibria, except at point B. Evolver simulations put the liquid fraction for point B at 0.0103. The dashed curve CBF represents unstable equilibrium asymmetric wet films, combinatorially the same as the wet cube cone. On segment BF, opposite films pass through each other in the central region, so this segment is not physical. The lower dashed curve DC represents equilibria reached by going downhill in energy from the symmetric unstable wet cone in the opposite direction from the central film. These films have the central region stretched vertically into a pillar rather than flattened into a slab. They are stable when 16-fold symmetry is imposed, but otherwise are unstable to jumping to vertical central films. This curve terminates at point D, the dry cone, because there is no equilibrium dry film with a fourfold junction as pictured in figure 7. This nonexistence follows from a calibration argument that the dry cube cone is the unique absolute minimum in a certain wedge, and that any equilibrium film in the wedge that is the graph of a function can be calibrated, and is hence an absolute minimum.

The details of the diagram around point B deserve a more detailed discussion. The central film configuration can be viewed as an overlapped wet cone with the overlaps pushed back. If an overlap is pushed back distance z, the work done may be approximated as follows: At internal pressure $-P$, the overlap is approximately spherical of radius $R = 2/P$. The plane of contact at pushback z will be approximately the area of a spherical slice, which is $A = 2\pi Rz$. The pushing force is $F = PA = 4\pi z$. The total work done pushing is $W = 2\pi z^2$. Since there are top and bottom films to push back, the total work done is really $W = 4\pi z^2$. Now suppose the wet cone has its unstable equilibrium at $z = z_0$, and its energy is $E(z) = E_0 - c(z - z_0)^2$. Because nothing unusual happens at point B if one lets the films interpenetrate, z_0 is a smooth function of liquid fraction. Evolver computations show that $c \approx 0.075$. The total energy of the central film configuration formed by taking some wet cone and pushing back will be

$$E_{net}(z) = E_0 - c(z - z_0)^2 + 4\pi z^2. \tag{14}$$

This has its minimum at

$$z = \frac{-2cz_0}{8\pi - 2c}. \tag{15}$$

This shows that curve AB does in fact meet DG where CF crosses, because $z = 0$ where $z_0 = 0$. Since z_0 is a nearly linear function of liquid fraction near point B, the central film line AB is nearly linear at B, and it does not go vertical. Furthermore, the eigenvalue for the central film does not approachs zero at B. This seems to be confirmed by the few Evolver tests I've done on it, although given all the perils of eigenvalue computing mentioned in this paper, I make the claim cautiously.

The diagram for constant pressure is very similar. The equilibria are the same, and the only difference is that the lines of constant pressure deviate slightly from the lines of constant volume (which are vertical in figure 6). For the same Plateau volume, the central film configuration has a slightly less negative internal pressure than the wet cone. At the dry extreme, this follows from the fact that the total length of the Plateau border in the dry cone is $8\sqrt{3} \approx 13.856$, while the length in the cube film (figure 1B) is about 13.746, according to the Evolver. At volume fraction 0.098 (near point B), the Evolver gives the pressure in the wet cone as -5.976, but with the central film -5.64.

The stability diagram for the full wet cube cone can best be visualized by taking the response space to be the two-dimensional eigenspace for the lowest eigenvalues found in section 4. The diagram contains three copies of figure 6 spread at 120° around a common axis DCE.

6 Other wet cones

By the results of Lamarle [L], Heppes [H], and Jean Taylor [T], there are eight possible dry cone singularities (not counting the triple junction of planes). The cone over the tetrahedron is stable when dry, and therefore when wet. We have seen the wet cube cone can be unstable. Of the other six cones, it turns out

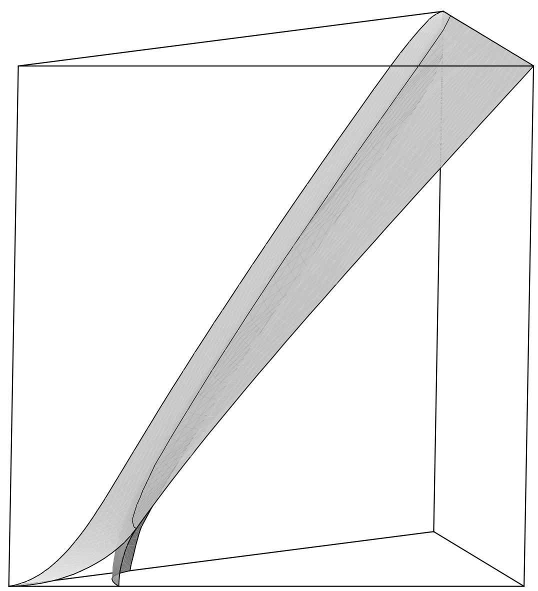

Figure 8: Fundamental region at the transition from a central film to a wet cone, point B in figure 6. The front plane of the bounding prism is the $y = 0$ symmetry plane, the back is $x = y$, the bottom is $z = 0$, the top is $z = 1$, and the right is $x = 1$.

from Evolver experiments that only the cone over the dodecahedron even has an equilibrium wet cone. For the five others, trying to form a wet cone results in a surface that has a gradient in the direction of the wet version of the minimal dry film. This does not happen with the cube and dodecahedron because their high symmetry rules out nonzero gradients.

By Evolver experiments of the same type as in section 4, the dodecahedral inner region turns out to be unstable, with negative eigenvalue eigenspaces of dimension 2 and 3, for a total of 5 unstable degrees of freedom. No doubt the full dodecahedral wet cone is unstable below some critical liquid fraction.

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